

A one-level limit order book model with memory and variable spread

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Abstract

Motivated by Cont and de Larrard (2013)'s seminal Limit Order Book (LOB) model, we propose a new model for the level I of a LOB in which the arrivals of orders and cancellations are still assumed to be mutually independent, memoryless, and stationary, but, unlike the aforementioned paper, the information about the standing orders at the opposite side of the book after each price change and the arrivals of new orders within the spread are incorporated. Our main result gives a diffusion approximation for the mid-price process, which sheds further light on the relation between the mid-price behavior at low frequencies and some LOB features not considered in earlier works. To illustrate the applicability of the proposed framework, we also develop a feasible method to compute several quantities of interest, such as the distribution of the time span between price changes and the probability of consecutive price increments conditioned on the current state of the book. These LOB model features are relevant in many applications such as high frequency trading and intraday risk management. The proposed method is also used to develop an efficient simulation scheme for the price dynamics, which is then applied to assess numerically the accuracy of the diffusion approximation.

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1. Introduction

Most modern financial exchanges make use of electronic communication networks that implement a continuous double auction trading mechanism. Two types of orders are available in these trading platforms: limit and market orders. Broadly, a bid (ask) limit order specifies a price at which a trader is willing to buy (sell) a determined number of shares of an asset, while a market order is a request to immediately buy or sell a specified number of shares at the best available prices out of all the active limit orders. Other than limit orders and market orders, cancellation of limit orders is another common operation. The Limit Order Book (LOB) aggregates all the outstanding limit orders at any given time and offers a unique glimpse into the forces and rules of price formation of an asset.

With the advent of real-time and historical LOB data, LOB modeling has received substantial attention in recent years. We refer to Gould et al. [9] for a nice survey of the extensive literature on the subject, which, among the most relevant to our work, includes Luckock [13], Kruk [12], Smith et al. [19], Rosu [17], Cont et al. [7], Abergel and Jedidi [1], Cont and Larrard [6], and Cont and Larrard [5]. The different models proposed so far obviously vary in complexity and detail depending on the applications being considered. In this work we present a new model for the dynamics of the best bid and ask levels of a LOB, which not only is able to incorporate several real features of LOB dynamics, but also is tractable enough for us to achieve two important objectives. Firstly, we are able to characterize the coarse-grain dynamics of the asset's price process by establishing a diffusion approximation for it. More concretely, we prove that the mid-price process of the asset, properly scaled in time and space, converges to a Brownian motion with drift. This type of scaling limit enables us to connect the features of the process at lower frequencies (say, minutes, hours, or days) to the statistical properties of events taking place at the millisecond scale. A classical application of this is to analyze the behavior of the asset's volatility as a function of both the intensities at which LOB events take place and a measure of the LOB's depth. Secondly, we are capable of computing several quantities of interest such as the distribution of the time span between price changes and the probability of consecutive price increments conditioned on the current state of the book. These LOB model features are relevant in many applications, but in particular, in high frequency algorithmic trading and intraday risk management.

Our main inspiration for the present work is drawn from [6]'s seminal work, where a Markovian model is proposed for the dynamics of the LOB's level I (i.e., the limit orders with the best prices to sell or buy the asset). There are two motivating factors for only considering the level I and not the entire book. Firstly, the asset's price is only determined by the level I and, secondly, the information contained in the level I is key for many high-frequency trading strategies and problems. By imposing a Poissonian order flow and some symmetry conditions on the shape of the order book, Cont and Larrard [6] established the diffusion approximation for the mid price process mentioned in the previous paragraph. Unfortunately, the results in [6] required several strong assumptions, the most important of which are:

- (i) a constant volume for all type of operations: market, limit, and cancellations;
- (ii) a constant spread of one tick between the best ask and bid prices at all times;
- (iii) constant parallel price shifts of one tick after each depletion of a level I queue;
- (iv) loss of “memory”, in the sense that, after each level I queue depletion, the information on the remaining limit orders at the side which was not depleted is reset.

The previous assumptions are, of course, idealizations. For instance, the assumption of constant spread is generally reasonable for “large” tick assets, in which the tick size is comparable

to a “typical” price increment. As illustrated in [6], for some “large tick” assets, the spread could be equal to one tick for more than 98% of the observations within a typical day. Nevertheless, in many situations, the constant spread assumption cannot be validated. Bouchaud et al. [3] argue that relatively large spreads may be created by monopolistic practices of market makers, high order processing costs, and large market orders. Pomponio and Abergel [16] (see also [15]) argue that traders keep track of the amounts of orders at the best quote in the LOB and typically restrict the size of their market orders to be less than these amounts. But, sometimes the speed of execution is more important than the market impact risk of large orders. In that case, orders larger than the size of the first limit (called trades-through) may be submitted. Pomponio and Abergel [16] argue that even though trade-throughs may rarely occur (with an occurrence probability of less than 5%), they make up for a non-negligible part of the daily-volume (up to 20% for the DAX index future). Since every trade-through widens the spread, a model that allows for variable spread is desirable.

In this work, we correct some of the shortcomings of the framework proposed in [6] as a way to account for the possibility of a variable spread, as described in the previous paragraph, and the fact that, in reality, the level I of the book is not completely reset after each price change. More concretely, while keeping some of the assumptions therein, such as one level at each side of the order book and a Markovian order flow with constant volume, our model keeps the information of the active orders on the other side of the book after either the best bid or ask price has changed. For instance, if the best bid queue gets depleted, the best bid price decreases one tick, but both the price and outstanding orders at the best ask are preserved. In order to avoid perpetual widening of the spread, we also allow the arrival of orders within the spread according to a Poisson process. We refer the reader to the main body of the paper for further details about our model.

As in [6], we establish a diffusive approximation for the price process, albeit using an essentially different method of proof. Our results build on the mathematical theory of countable positive recurrent Harris chains (see, e.g., [14]). Furthermore, to illustrate the applicability of our framework, we put forward a feasible procedure to compute several quantities of interest, conditioned on the initial state of the book. Some examples of these include the distribution of the duration between price changes, the probability of a price increase, and the probability of two consecutive increments on the price. The reader is referred to Cont et al. [7] for applications of these quantities to high-frequency trading. The main tool for the derived formulas is an explicit characterization of the joint distribution of the time of a depletion at the level I and the amount of orders at the remaining queue at such a time. Such a characterization depends on the eigenvalues and eigenvectors of the generator of a suitable two-dimensional random walk. The developed method is also applied to devise an efficient simulation algorithm for the dynamics of the LOB, which is subsequently used to numerically study the convergence of the midprice process towards its diffusive limiting process.

Let us finish this introduction with a brief discussion about the connection of our work with some earlier literature. Abergel and Jedidi [1] also obtain a diffusive approximation for the mid-price process; however, our model cannot be framed within their approach since the spread in their model remains bounded, while, in our model, it can take values on $\mathbb{Z}_+$. Let us also remark that our model can still account for the empirical observation of Cont and Larrard [6] that the spread spends a large amount of time at the value of 1 by setting a large value for the intensity of arrivals of limit orders within the spread. This feature is, however, not possible to incorporate in [1]’s model since the intensity of arrivals at the first potential bid and ask price level is constant, whether this level is already occupied by limit orders or not. Another relevant work is Cont and Larrard [5], in which the assumptions of constant volume and absence of memory of Cont and

Larrard [6] (assumptions (i) and (iv) as described above) are relaxed, but they do not establish a diffusive approximation for the price process (see [Remark 2.1](#) for further comparisons).

The present article is organized as follows. Section 2 introduces the model which allows variable spread and obtains a Functional Central Limit Theorem for the midprice process. Section 3 introduces a method to efficiently compute several quantities of interest related to the LOB. In Section 4, we analyze, via simulation, the rate of convergence of the midprice process to its diffusive process as well as the behavior of the spread and the asymptotic volatility in relation to the different model's parameters. To this end, we develop an “efficient” method to simulate the dynamics of the LOB level I based on our results of Section 3. In Section 4, we also compute numerically some of the quantities of interest considered in Section 3 and study their behaviors under both our assumptions and those in [6]. Section 5 summarizes our main conclusions. Finally, some of the technical proofs are presented in the [Appendix](#).

2. A one-level LOB model with memory and variable spread

In this section, we introduce a new framework for the dynamics of the level I of a LOB, whose characteristics better mimic reality as compare to those in the model introduced in [6]. Our main result is a diffusive approximation for the mid-price process. More specifically, if $\{s_t\}_{t \geq 0}$ denotes the mid-price process of the stock, then, for some appropriate constants $\sigma > 0$ and $m \in \mathbb{R}$, the following invariance principle holds:

$$\frac{s_{nt} - nmt}{\sqrt{n}} \Rightarrow \sigma W_t, \quad n \rightarrow \infty, \quad (1)$$

where $\{W_t\}_{t \geq 0}$ is a Wiener process and, hereafter, $\Rightarrow$ denotes convergence in distribution. Heuristically, if we think of $1/n$ as the time scale at which the process is observed, (1) says that the price process can be well approximated by a Brownian motion with drift at “small” scales (typically, 10 or more seconds, depending on the speed of the book events, which typically happen at the order of milliseconds).

This section is organized as follows. We first introduce the model and necessary notation in Section 2.1. Section 2.2 proves a Law of Large Numbers for the interarrival times between price changes, which in turn is needed to determine the appropriate time scaling of the price process. Finally, we proceed to obtain a Functional Central Limit Theorem (FCLT) for the price process itself in Section 2.3.

2.1. LOB dynamics

As mentioned in the introduction, a bid (ask) limit order specifies a price at which a trader is willing to buy (sell) a determined number of shares of an asset, while a market order is a request to immediately buy or sell a specified number of shares at the best available prices out of all the active limit orders. The Limit Order Book (LOB) aggregates all the active buy or sell limit orders at any given time. Conceptually, an LOB can be visualized as a system of (possibly empty) FIFO (first-in-first-out) queues (one for each possible tick price). [Fig. 1](#) gives a graphical representation of a LOB. The lowest price of all the active ask limit orders is called the ask price and is the best available price to buy the asset with a market order. Similarly, the highest price of all the outstanding bid limit orders is called the bid price and is the best available price to sell the asset with a market order. In this fashion, this trading mechanism matches sell and buy market orders with orders sitting at the best bid and ask prices, respectively. The separation between

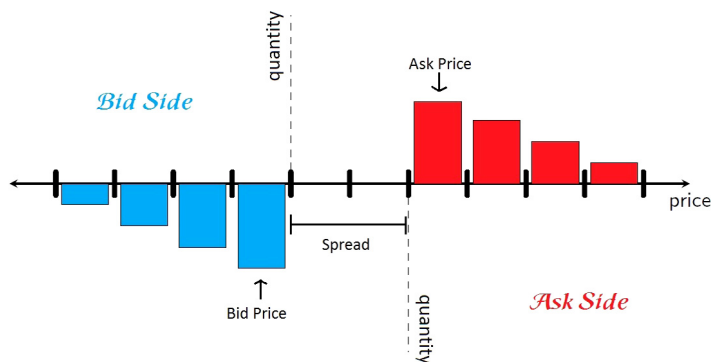


Fig. 1. Graphical representation of a Limit Order Book. The Bid Limit Orders (to the left) are displayed in blue, while the Ask Limit Orders (to the right) are displayed in red. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the ask and the bid prices is called the spread of the LOB, whereas the queues corresponding to the best bid and ask prices are called the level I of the book. Other than limit orders and market orders, cancellation of limit orders is another common operation, which accounts for a considerably large fraction of the events in an order book (cf. Harris [10]). Note that a price change would occur in either of two situations: one of the two queues at the level I gets depleted (due to the cancellation of limit orders or the arrival of a market order) or a new limit order is posted within the spread. More specifically, in the first situation, the best bid (ask) price would decrease (increase) if the queue of limit orders at the best bid (ask) got depleted. In that case, the next nonempty bid (ask) queue would become part of the new level I. In the second situation, the best bid (ask) price would increase (decrease) if a new bid (ask) limit order were posted within the spread, if possible.

In this work, we only consider the level I of the order book, which, as previously explained, suffices to determine the evolution of the price process of the asset. As in [6], we also assume constant price shifts of one tick after each price change. For example, suppose that the best bid price is at S^b and that its corresponding queue gets depleted after the arrival of a market order or a cancellation. Then, if we denote δ the tick size, the new best bid queue is set at the price $S^b - \delta$, hence causing the spread to widen. We assume that the size of this new queue of limit orders is generated from a distribution f^b on $\mathbb{Z}_+$, independently of any other information on the LOB, while the amount and position of the limit orders at the best ask level are kept unchanged. Similarly, if the queue at the best ask price gets depleted after the arrival of a market order or cancellation, then a new queue is generated at the price $S^a + \delta$, where S^a is the best ask price prior to the arrival of the order. The size of the new queue at the best ask is assumed to be generated from a distribution f^a on $\mathbb{Z}_+$, independently of any other information. In that case, we assume that the bid side of the book remains unchanged.

The distributions f^a and f^b are meant to reflect the stationary behavior of the queue sizes at the next best queues after depletion. Throughout, we assume that both distributions f^a and f^b are supported on $\{1, 2, \dots, N^*\}$, for some fixed $N^* \in \mathbb{Z}_+$, which can be chosen arbitrarily large. This simplifying assumption is imposed in order to guarantee the recurrence of the underlying Markov chain driving the dynamics of the price process. Also, for simplicity, the tick is set to be $\delta = 1$.

When the spread is more than 1, there is also the possibility of a price change due to the arrival of a new set of orders within the spread at either the ask or bid side of the LOB. In the

former case, the best ask price decreases by δ , while the bid side remains unchanged. In the latter case, the best bid price increases by δ , while the other side of the order book does not change. As before, the size of a new queue of limit orders is generated from the distribution f^a or f^b , independently of any other variables, depending on whether the new set of limit orders is at the ask or bid side.¹

We now proceed to give a formal mathematical formulation of the LOB dynamics. To that end, we need some notation:

- (i) In what follows, ζ_0 denotes the initial spread, while ζ_i , for $i \geq 1$, denotes the spread after the i th price change. The sizes of the best ask and bid queues at time t are denoted by q_t^a and q_t^b , respectively. Also, for $i \geq 1$, τ_i represents the time span between the $(i-1)$ th and i th price changes and we set $\tau_0 = 0$.
- (ii) Throughout, $\{\hat{Y}^{a,i}\}_{i \geq 0}$ and $\{\hat{Y}^{b,i}\}_{i \geq 0}$ are independent sequences of i.i.d. random variables, taking values on $\Omega_{N^*} := \{1, 2, \dots, N^*\}$, and with respective distributions f^a and f^b . These will indicate the amount of orders at the best ask or bid queues after that particular side changes in price.
- (iii) Let $\{L_i^a(\zeta)\}_{i \geq 0, \zeta \in \mathbb{Z}_+}$ and $\{L_i^b(\zeta)\}_{i \geq 0, \zeta \in \mathbb{Z}_+}$ be independent sequences of independent random variables such that, for every $i \in \mathbb{N}^+$, $L_i^a(\zeta)$ and $L_i^b(\zeta)$ are exponentially distributed with parameter $\alpha \mathbb{1}_{\{\zeta > 1\}}$.² These variables are also independent of any other variables in the system. Hereafter, $L_i(\zeta) := L_i^a(\zeta) \wedge L_i^b(\zeta)$. We shall interpret $L_i^a(\zeta)$ and $L_i^b(\zeta)$ as the times for a new set of orders to arrive at the ask and bid side, respectively, after the i th-price change, when the spread is at the value ζ .
- (iv) For any starting point $x \in \bar{\Omega}_{N^*} := \{0, 1, \dots, N^*\}$, let $Q(x) := \{Q_t(x)\}_{t \geq 0}$ be a continuous time Markov process with state space $\bar{\Omega}_{N^*}$ such that $Q_0(x) = x$ and its transition matrix $Q : \bar{\Omega}_{N^*} \times \bar{\Omega}_{N^*} \rightarrow \mathbb{R}$ is given by:

$$\begin{aligned} Q_{j,j+1} &= \lambda, & \text{for } 0 \leq j \leq N^* - 1, & & Q_{j,j-1} &= \nu, & \text{for } 1 \leq j \leq N^*, \\ Q_{j,j} &= -(\nu + \lambda), & \text{for } 1 \leq j \leq N^* - 1, & & Q_{N^*,N^*} &= -\nu, \\ Q_{j,\ell} &= 0, & \text{otherwise,} \end{aligned} \quad (2)$$

where $\nu := \mu + \theta$ and λ, μ , and θ are interpreted as the intensity of arrivals of limit orders at the level I, market orders, and cancellations, respectively.

- (v) Finally, for any $i \geq 0$ and $x \in \bar{\Omega}_{N^*}$, we let $Q^{a,i}(x) := \{Q_t^{a,i}(x)\}_{t \geq 0}$ and $Q^{b,i}(x) := \{Q_t^{b,i}(x)\}_{t \geq 0}$ be processes such that

$$Q^{a,i}(x) \stackrel{\mathcal{D}}{=} Q^{b,i}(x) \stackrel{\mathcal{D}}{=} Q(x), \quad (3)$$

and the collection of processes $\{Q^{a,i}(x), Q^{b,i}(x)\}_{i \geq 0, x \in \bar{\Omega}_{N^*}}$ are mutually independent, and also independent of the processes introduced in the points (ii)–(iii).

We are ready to give a formal construction of the LOB dynamics. Fix $\tau_0 = 0$ and define the processes

$$X_t^{a,0} := Q_t^{a,0}(x_0^a), \quad X_t^{b,0} := Q_t^{b,0}(x_0^b), \quad (4)$$

¹ Our results are still valid if one takes these distributions to be different from the one used when a level I queue gets depleted.

² In particular, $L_i(1) = M_i(1) = \infty$, a.s., for all i .

for some arbitrary, possibly random, initial queue sizes $(x_0^a, x_0^b) \in \Omega_{N^*}^2$, which are also assumed to be independent of any of the other processes considered in the points (i)–(v) above. With the notation

$$\sigma^{a,1} := \inf \left\{ t \geq 0 : X_t^{a,0} = 0 \right\} \wedge L_0^a(\zeta_0), \quad \sigma^{b,1} := \inf \left\{ t \geq 0 : X_t^{b,0} = 0 \right\} \wedge L_0^b(\zeta_0),$$

at hand, the time of the first price change can now be defined by

$$T_1 := \tau_1 := \sigma^{a,1} \wedge \sigma^{b,1}, \quad (5)$$

while, for $t \in [0, T_1)$, the queue sizes at the best ask and bid prices are respectively given by

$$q_t^a = X_t^{a,0}, \quad q_t^b = X_t^{b,0}.$$

The number of orders at each side of the LOB and the spread at time T_1 are then set as

$$\begin{aligned} q_{T_1}^a &:= x_1^a := \hat{Y}^{a,1} \mathbb{1}_{\{\tau_1 = \sigma^{a,1}\}} + X_{\tau_1}^{a,0} \mathbb{1}_{\{\tau_1 = \sigma^{b,1}\}}, \\ q_{T_1}^b &:= x_1^b := \hat{Y}^{b,1} \mathbb{1}_{\{\tau_1 = \sigma^{b,1}\}} + X_{\tau_1}^{b,0} \mathbb{1}_{\{\tau_1 = \sigma^{a,1}\}}, \\ \zeta_1 &:= \zeta_0 + \mathbb{1}_{\{\tau_1 < L_0^a(\zeta_0) \wedge L_0^b(\zeta_0)\}} - \mathbb{1}_{\{\tau_1 = L_0^a(\zeta_0)\}} - \mathbb{1}_{\{\tau_1 = L_0^b(\zeta_0)\}}. \end{aligned}$$

This process is continued recursively. Concretely, for $i \geq 1$, we set

$$\begin{aligned} q_t^a &:= X_{t-T_i}^{a,i}, \quad q_t^b := X_{t-T_i}^{b,i}, \quad \text{for } t \in [T_i, T_{i+1}), \quad T_{i+1} := T_i + \tau_{i+1}, \\ \tau_{i+1} &:= \sigma^{a,i+1} \wedge \sigma^{b,i+1}, \\ \zeta_{i+1} &:= \zeta_i + \mathbb{1}_{\{\tau_{i+1} < L_i^a(\zeta_i) \wedge L_i^b(\zeta_i)\}} - \mathbb{1}_{\{\tau_{i+1} = L_i^a(\zeta_i)\}} - \mathbb{1}_{\{\tau_{i+1} = L_i^b(\zeta_i)\}} \end{aligned}$$

where

$$\begin{aligned} X_t^{a,i} &= Q_t^{a,i}(x_i^a), \quad X_t^{b,i} = Q_t^{b,i}(x_i^b), \\ \sigma^{a,i+1} &= \inf \left\{ t > 0 : X_t^{a,i} = 0 \right\} \wedge L_i^a(\zeta_i), \\ \sigma^{b,i+1} &= \inf \left\{ t > 0 : X_t^{b,i} = 0 \right\} \wedge L_i^b(\zeta_i), \\ x_{i+1}^a &:= \hat{Y}^{a,i+1} \mathbb{1}_{\{\tau_{i+1} = \sigma^{a,i+1}\}} + X_{\tau_{i+1}}^{a,i} \mathbb{1}_{\{\tau_{i+1} = \sigma^{b,i+1}\}}, \\ x_{i+1}^b &:= \hat{Y}^{b,i+1} \mathbb{1}_{\{\tau_{i+1} = \sigma^{b,i+1}\}} + X_{\tau_{i+1}}^{b,i} \mathbb{1}_{\{\tau_{i+1} = \sigma^{a,i+1}\}}. \end{aligned}$$

The above formulation justifies the following identities:

$$\begin{aligned} \mathbb{P}((\tilde{x}_k, \zeta_k, \tau_k) \in B \times C \times D | (\tilde{x}_{k-1}, \zeta_{k-1}) = (\tilde{x}, \zeta)) \\ = \mathbb{P}((\tilde{x}_1, \zeta_1, \tau_1) \in B \times C \times D | (\tilde{x}_0, \zeta_0) = (\tilde{x}, \zeta)), \end{aligned} \quad (6)$$

$$\begin{aligned} \mathbb{P}\left((\tilde{x}_k, \zeta_k, \tau_k) \in B \times C \times D | \{(\tilde{x}_i, \zeta_i, \tau_i)\}_{i=0}^{k-1}\right) \\ = \mathbb{P}((\tilde{x}_k, \zeta_k, \tau_k) \in B \times C \times D | (\tilde{x}_{k-1}, \zeta_{k-1})), \end{aligned} \quad (7)$$

where $\tilde{x}_k := (x_k^b, x_k^a)$, and x_k^b and x_k^a respectively represent the sizes at the best bid and ask queues after the k th prices change. In particular, it follows that

$$\tau_k \perp_{\{(\tilde{x}_{i-1}, \zeta_{i-1})\}_{i \geq 0}} (\tau_{k-1}, \dots, \tau_1), \quad k \geq 2. \quad (8)$$

The above identity also implies the mutual independence of $\{\tau_1, \dots, \tau_n\}$ given $\{(\tilde{x}_{i-1}, \zeta_{i-1})\}_{i \geq 0}$. Furthermore, it is easy to see that the process

$$(\Xi_t, \Upsilon_t) := \sum_{k=0}^n (\tilde{x}_k, \zeta_k) \mathbb{1}_{[T_k, T_{k+1})}(t)$$

is semimarkov in the sense of Çinlar [4].

Remark 2.1. As mentioned in the introduction, one of the key features of the model proposed above is the incorporation of memory. Other recent works have also considered this feature. Notably, Cont and Larrard [5] assume that the level I queue sizes after each price change, q_{T_i} , is a function $g(q_{T_i^-}, \varepsilon_i)$ of the level I queue sizes before the price change, $q_{T_i^-}$, and a sequence of i.i.d. random innovations $\{\varepsilon_i\}_{i \geq 1} := \{(\varepsilon_i^b, \varepsilon_i^a)\}_{i \geq 1}$. One of the examples considered therein is the case of “pegged limit orders” in which $q_{T_i}^a = \beta q_{T_i^-}^a + \varepsilon_i^a$ and $q_{T_i}^b = \varepsilon_i^b$, when the best bid queue gets depleted (with a similar relation holding for the case when the best ask queue gets depleted). Here, $\beta \in [0, 1]$ is a constant and $(\varepsilon_i^b, \varepsilon_i^a)$ have a distribution F in $\mathbb{Z}_+^2$. However, Assumption 3 in [5] precludes the situation where $\varepsilon_i^a = 0$, a.s., and $\beta = 1$, which is the type of memory we consider in this work. Let us also remark that Cont and Larrard [5] do not establish a diffusive approximation for the price process, but rather, a heavy traffic approximation for the queue sizes of the LOB level I as a Markovian jump–diffusion process on the first quadrant. On the interior of the first quadrant, the process follows a planar Brownian motion with some given drift and covariance matrix. Every time that the process hits one of the axis, this is then shifted to the interior of the quadrant (according to some rules). Therefore, as described in Proposition 1 in [6], the price process can be approximated by a random walk, which moves one tick to the left or right depending on whether the just described jump–diffusion process hits the x - or y -axis. Note that this is not the same type of continuous diffusive approximation for the price process as the one considered in this work.

2.2. A law of large numbers for the modified interarrival times

Our first ingredient towards (1) is to establish a law of large numbers (LLN) for the time of the n th-price change, $T_n = \sum_{k=1}^n \tau_k$, using ergodic results for Markov chains. To that end, we first introduce some needed notation. Let $Z = \{Z_t\}_{t \in \mathbb{N}}$ denote a Markov chain on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with countable state space Ξ and transition probability matrix $P : \Xi \times \Xi \rightarrow [0, 1]$. For any probability measure $\mu = \{\mu(\hat{y}), \hat{y} \in \Xi\}$ on Ξ , $\hat{y} \in \Xi$, and $A \subset \Xi^{\mathbb{Z}_+} = \{(z_1, z_2, \dots) \mid z_i \in \Xi\}$, we define

$$\mathbb{P}_{\hat{y}}(Z \in A) := \mathbb{P}(Z \in A \mid Z_0 = \hat{y}), \quad \mathbb{P}_{\mu}(Z \in A) := \sum_{\hat{y} \in \Xi} \mu(\hat{y}) \mathbb{P}_{\hat{y}}(Z \in A).$$

As usual, $\mathbb{E}_{\mu}$ denotes the expectation with respect to the probability measure $\mathbb{P}_{\mu}$. We say that an event A occurs $\mathbb{P}_*$ -a.s. if A occurs $\mathbb{P}_{\hat{y}}$ -a.s. for all $\hat{y} \in \Xi$.

Let us recall that an irreducible Markov chain on a countable state space Ξ is either transient or recurrent, while a set A is called Harris recurrent if

$$\mathbb{P}_z \left(\sum_{n=1}^{\infty} \mathbb{1}_{\{Z_n \in A\}} = \infty \right) = 1, \quad z \in A.$$

A Markov chain is called Harris recurrent if it is irreducible and every set A is Harris recurrent. Also, the Markov chain Z is called positive if it is irreducible and admits an invariant probability measure, while a positive and Harris recurrent chain is called positive Harris (cf. Chapter 10 [14]). The following result from [14] (Theorem 17.0.1 therein) is key to obtain the aforementioned LLN.

Theorem 2.2. *Suppose that $\{Z_t\}_{t \in \mathbb{N}}$ is a positive Harris chain with invariant probability measure π . Then, for any g satisfying $\pi(|g|) := \sum_x \pi(x)|g(x)| < \infty$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n g(Z_t) = \pi(g), \quad \mathbb{P}_*-a.s.$$

In the sequel, we shall use Theorem 2.2 to show a LLN for $T_n = \sum_{i=1}^n \tau_i$ by expressing each τ_i in terms of an appropriate Markov chain $Z := \{Z_t\}_{t \in \mathbb{N}}$. Concretely, throughout the remaining of this subsection, we take

$$Z := \{Z_t\}_{t \geq 1} := \{(\tilde{x}_{t-1}, \zeta_{t-1}, \tilde{x}_t, \zeta_t)\}_{t \geq 1}, \quad (9)$$

where we recall that $\tilde{x}_t := (x_t^b, x_t^a)$ and ζ_t respectively represent the number of orders at the book's level I (bid and ask) and the spread after the t th price change (see Section 2.1 for details about the notation). By (7), we can see that Z is a Markov chain with countable state space

$$\begin{aligned} \Xi := & \left\{ (y_1, c_1, y_2, c_2) \mid y_1 = (y_1^a, y_1^b) \in \Omega_{N^*}^2, \right. \\ & \left. y_2 = (y_2^a, y_2^b) \in \Omega_{N^*}^2, c_1, c_2 \in \mathbb{Z}_+, |c_1 - c_2| = 1 \right\}. \end{aligned}$$

Furthermore, fixing $U_n := (\tilde{x}_n, \zeta_n)$ and noting that $U := \{U_n\}_{n \geq 0}$ is itself a Markov chain by (6)–(7), it follows that

$$\begin{aligned} P(\hat{y}, \hat{z}) := & \mathbb{P}(Z_n = \hat{z} \mid Z_{n-1} = \hat{y}) = \mathbb{P}((U_{n-1}, U_n) = \hat{z} \mid (U_{n-2}, U_{n-1}) = \hat{y}) \\ & = \mathbb{P}(U_n = (z_2, d_2) \mid U_{n-1} = (z_1, d_1)), \end{aligned}$$

where $\hat{y} := (y_1, c_1, y_2, c_2) \in \Xi$ and $\hat{z} := (z_1, d_1, z_2, d_2) \in \Xi$ with $(y_2, c_2) = (z_1, d_1)$.

Our first objective is to prove that we can apply Theorem 2.2 to the chain Z introduced in (9). Since, for a countable state Markov chain, irreducibility reduces to see that all states communicate to one another, by the description of the dynamics of Z given in the previous section, Z is clearly irreducible. Moreover, the Markov chain Z has an invariant probability measure provided that Z is positive recurrent (cf. [2, Asmussen, Corollary I.3.6]). Furthermore, in that case, Z will also be positive Harris chain, since, for a countable-state Markov chain, Harris recurrence is equivalent to plain recurrence (see the discussion below Theorem 9.0.1 in [14]). So, it remains to prove that Z is indeed positive recurrent, as stated by the following result.

Theorem 2.3. *If $\alpha \geq \mu + \theta$, then the Markov chain $Z := \{Z_t\}_{t \geq 1} := \{(\tilde{x}_{t-1}, \zeta_{t-1}, \tilde{x}_t, \zeta_t)\}_{t \geq 1}$ is positive recurrent.*

Before proving the previous result, we state some needed results and lemmas. A well-known sufficient condition for a Markov chain to be positive recurrent over a countable state space is given by the following so-called Foster or mean drift conditions (cf. Theorem I.5.3 in

Asmussen [2]) for some function $h : \Xi \rightarrow \mathbb{R}$, a constant $\epsilon > 0$, and a finite set $F \subset \Xi$:

$$\begin{aligned} \text{(i)} \quad & \inf_{\hat{z} \in \Xi} h(\hat{z}) > -\infty, & \text{(ii)} \quad & \sum_{\hat{z} \in \Xi} P(\hat{y}, \hat{z})h(\hat{z}) < \infty, \quad \hat{y} \in F, \\ \text{(iii)} \quad & \sum_{\hat{z} \in \Xi} P(\hat{y}, \hat{z})h(\hat{z}) < h(\hat{y}) - \epsilon, \quad \hat{y} \notin F. \end{aligned} \quad (10)$$

In order to verify that Z satisfies the previous conditions, we need two preliminary results. The following result constructs a super-harmonic function φ , outside the set $F_0 := \{\hat{y} \in \Xi : \hat{y} = (y_1, 2, y_2, 1)\}$. Recall that φ is said to be a super-harmonic function (cf. Section 17.1.2 in [14]) at some $\hat{y} \in \Xi$ if

$$(P\varphi)(\hat{y}) := \sum_{\hat{z} \in \Xi} P(\hat{y}, \hat{z})\varphi(\hat{z}) \leq \varphi(\hat{y}). \quad (11)$$

The proof of the next result is deferred to [Appendix](#).

Lemma 2.4. *Under the notation in Section 2.1, let $\varsigma(x) := \inf\{t > 0 : Q_t^{a,0}(x_1) \wedge Q_t^{b,0}(x_2) = 0\}$, for $x := (x_1, x_2)$, and let $L := L_1^a(2) \wedge L_1^b(2)$ (i.e., L is exponentially distributed with rate 2α). Also, for any $\hat{y} = (y_1, j \pm 1, y_2, j) \in \Xi$, let $\varphi(\hat{y}) : \Xi \rightarrow \mathbb{R}$ be given by*

$$\begin{aligned} \varphi((y_1, j \pm 1, y_2, j)) &:= \varphi(j) \\ &:= \begin{cases} \left(\frac{1 + \sqrt{1 - 4p_1(1 - p_{N^*})}}{2p_1} \right)^j & \text{if } p_1(1 - p_{N^*}) < \frac{1}{4}, \\ \left(\frac{1}{2p_1} \right)^j & \text{if } p_1(1 - p_{N^*}) = \frac{1}{4}, \\ \left(\frac{1 - p_{N^*}}{p_1} \right)^{\frac{j}{2}} \cos(j\theta) & \text{if } p_1(1 - p_{N^*}) > \frac{1}{4}, \end{cases} \end{aligned} \quad (12)$$

where

$$\begin{aligned} p_1 &:= \mathbb{P}(L > \varsigma((1, 1))), & p_{N^*} &:= \mathbb{P}(L > \varsigma((N^*, N^*))), \\ \theta &:= \arctan(\sqrt{4p_1(1 - p_{N^*}) - 1}), \end{aligned}$$

Then, φ is a super-harmonic function for the process Z given in (9), at any $\hat{y} \in \Xi \setminus F_0$, where $F_0 := \{\hat{y} \in \Xi : \hat{y} = (y_1, 2, y_2, 1)\}$.

The next result is crucial to construct the function h satisfying the conditions in (10). Its proof is also deferred to [Appendix](#).

Lemma 2.5. *Using the notation of Lemma 2.4, for any $\alpha \geq \mu + \theta$, it holds that $\lim_{j \rightarrow \infty} \varphi(j) = \infty$.*

Finally, we can prove that the Markov chain Z is positive recurrent.

Proof of 2.3. Consider the function $\varphi(\hat{y}) = \varphi((y_1, j \pm 1, y_2, j)) = \varphi(j)$ as given by Eq. (12). Then, by the proof of Lemma 2.4, we know that,

$$\varphi(j-1)(1 - p_{N^*}) + \varphi(j+1)p_1 = \varphi(j). \quad (13)$$

Take any $\epsilon \in (0, 1)$ and define $h(\hat{y}) = \varphi(\hat{y}) - \epsilon/(p_1 - p_{N^*})$, and $h(j) = h(\hat{y})$, for any $\hat{y} = (y_1, j \pm 1, y_2, j)$. Notice that $p_1 > p_{N^*}$ and, by Lemma 2.5, $h(j) \rightarrow \infty$ as $j \rightarrow \infty$.

Let $\Theta \in \mathbb{R}$ be such that for $j > \Theta$, we have that $\varphi(j) > \epsilon/(p_1 - p_{\mathbf{N}^*})$, and let also $F = \{\hat{y} \in \Xi : \hat{y} = (y_1, z_1, y_2, z_2), z_2 \leq \Theta + 1\}$. Notice that F is a finite set. From the definition of h and the fact that $P(\hat{y}, \hat{z}) > 0$ for only finitely many $\hat{z}$, it is clear that h satisfies the first two Foster conditions shown in (10). On the other hand, following similar steps as in the proof of Lemma 2.4, we have that, for every $\hat{y} \notin F$,

$$\begin{aligned} \sum_{\hat{z} \in \Xi} P(\hat{y}, \hat{z})h(\hat{z}) &= h(j-1)\mathbb{P}(N < \zeta(y_2)) + h(j+1)\mathbb{P}(\zeta(y_2) < N) \\ &= (\varphi(j-1) - \epsilon/(p_1 - p_{\mathbf{N}^*}))\mathbb{P}(N < \zeta(y_2)) \\ &\quad + (\varphi(j+1) - \epsilon/(p_1 - p_{\mathbf{N}^*}))\mathbb{P}(\zeta(y_2) < N) \\ &\leq (\varphi(j-1) - \epsilon/(p_1 - p_{\mathbf{N}^*}))(1 - p_{\mathbf{N}^*}) + (\varphi(j+1) - \epsilon/(p_1 - p_{\mathbf{N}^*}))p_1 \\ &= \varphi(j) - \epsilon(1 - p_{\mathbf{N}^*} + p_1)/(p_1 - p_{\mathbf{N}^*}) \\ &= h(j) - \epsilon = h(\hat{y}) - \epsilon. \end{aligned}$$

This proves the last Foster condition given in (10) and the fact that Z is positive recurrent follows. $\square$

Once we have proved that Z satisfies the hypothesis of Theorem 2.2, we now introduce the functions on which the theorem is applied. For any $\hat{x} = (x_0, c_0, x_1, c_1) \in \Xi$, let

$$\begin{aligned} f(\hat{x}) &:= \mathbb{E}(\tau_1 | \hat{x}) := \mathbb{E}(\tau_1 | (\tilde{x}_0, \zeta_0) = (x_0, c_0), (\tilde{x}_1, \zeta_1) = (x_1, c_1)), \\ g_t(\hat{x}) &:= \mathbb{P}(\tau_1 > t | \hat{x}) := \mathbb{P}(\tau_1 > t | (\tilde{x}_0, \zeta_0) = (x_0, c_0), (\tilde{x}_1, \zeta_1) = (x_1, c_1)). \end{aligned}$$

We have the following result, whose proof is deferred to the Appendix:

Lemma 2.6. Suppose that the conditions of Theorem 2.3 hold and let π be the invariance probability of the chain Z . Then, P_* -a.s.,

$$(i) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(Z_k) = \mathbb{E}_\pi(\tau_1), \quad (ii) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n g_t(Z_k) = \mathbb{P}_\pi(\tau_1 > t). \quad (14)$$

In order to obtain the LLN for the interarrival times $\{\tau_i\}_{i \geq 1}$, we shall show that the Laplace transform of the random variables $T_n = \tau_1 + \dots + \tau_n$, properly scaled, converges to the Laplace transform of a random variable T , for which we need the following:

Proposition 2.7. For $u \in \mathbb{R}_+$ and $\hat{x} = (x_0, c_0, x_1, c_1) \in \Xi$, define the functions

$$\begin{aligned} G(u | \hat{x}) &:= \mathbb{E}(e^{-u\tau_1} | (\tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1) = (x_0, c_0, x_1, c_1)), \\ \kappa(\hat{x}) &:= \mathbb{E}(-\tau_1 | (\tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1) = (x_0, c_0, x_1, c_1)). \end{aligned}$$

Then, under the assumption of Lemma 2.5, for any $u \in [0, \infty]$,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \ln G\left(\frac{u}{n} \middle| \tilde{x}_{k-1}, \zeta_{k-1}, \tilde{x}_k, \zeta_k\right) = -u\mathbb{E}_\pi(\tau_1), \quad P_*\text{-a.s.}, \quad (15)$$

where π is the stationary measure of the Markov chain $\{Z_n\}_{n \geq 0}$.

Proof. First note that the statement is trivial for $u = 0$. By (47), $\tau_1 < \infty$ a.s., thus, $\mathbb{E}(e^{-u\tau_1} | \tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1) > 0$ a.s. Assume now that $u \in (0, \infty)$, then, by Jensen's inequality,

$$\frac{1}{u} \ln G(u | \tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1) \geq \frac{1}{u} \mathbb{E}(\ln e^{-u\tau_1} | \tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1) = \kappa(\tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1).$$

Therefore, $\ln G\left(\frac{u}{n} \middle| \tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1\right) \geq \frac{u}{n} \kappa(\tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1)$ and, thus,

$$\sum_{k=1}^n \ln G\left(\frac{u}{n} \middle| \tilde{x}_{k-1}, \zeta_{k-1}, \tilde{x}_k, \zeta_k\right) \geq \frac{u}{n} \sum_{k=1}^n \kappa(\tilde{x}_{k-1}, \zeta_{k-1}, \tilde{x}_k, \zeta_k), \quad (16)$$

which, by Eq. (14-i), implies that,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \sum_{k=1}^n \ln G\left(\frac{u}{n} \middle| \tilde{x}_{k-1}, \zeta_{k-1}, \tilde{x}_k, \zeta_k\right) &\geq \liminf_{n \rightarrow \infty} \frac{u}{n} \sum_{k=1}^n \kappa(\tilde{x}_{k-1}, \zeta_{k-1}, \tilde{x}_k, \zeta_k) \\ &= -u \mathbb{E}_\pi(\tau_1) \quad P_*\text{-a.s.} \end{aligned}$$

Next, note that

$$\begin{aligned} \frac{1}{u} \ln G(u \middle| \tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1) &\leq \mathbb{E}\left(\frac{e^{-u\tau_1} - 1}{u} \middle| \tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1\right) \\ &= \int_0^\infty -e^{-ut} \mathbb{P}[\tau_1 > t \mid \tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1] dt, \end{aligned}$$

where for the first inequality we used that $\ln(x) \leq x - 1$, for $x > 0$, and for the last equality we used the identity $\mathbb{E}(g(X)) = g(0) + \int_0^\infty g'(t) P[X > t] dt$, which is valid for any positive random variable X and monotonic differentiable function $g : [0, \infty) \rightarrow \mathbb{R}$. Therefore, we have that:

$$\ln G\left(\frac{u}{n} \middle| \tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1\right) \leq \frac{u}{n} \int_0^\infty -e^{-\frac{u}{n}t} \mathbb{P}[\tau_1 > t \mid \tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1] dt.$$

This last inequality, Fatou's Lemma, and Eq. (14-i) yield,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sum_{k=1}^n \ln G\left(\frac{u}{n} \middle| \tilde{x}_{k-1}, \zeta_{k-1}, \tilde{x}_k, \zeta_k\right) &\leq \limsup_{n \rightarrow \infty} \sum_{k=1}^n -\frac{u}{n} \int_0^\infty e^{-\frac{u}{n}t} \mathbb{P}(\tau_1 > t \mid \tilde{x}_{k-1}, \zeta_{k-1}, \tilde{x}_k, \zeta_k) dt \\ &\leq \int_0^\infty \limsup_{n \rightarrow \infty} \left(-ue^{-\frac{u}{n}t}\right) \left(\frac{1}{n} \sum_{k=1}^n \mathbb{P}(\tau_1 > t \mid \tilde{x}_{k-1}, \zeta_{k-1}, \tilde{x}_k, \zeta_k)\right) dt \\ &= \int_0^\infty -u \mathbb{P}_\pi(\tau_1 > t) dt = -u \mathbb{E}_\pi(\tau_1), \quad P_*\text{-a.s.} \end{aligned} \quad (17)$$

Together (16) and (17) imply (15). $\square$

We are now ready to show the main result of this section.

Theorem 2.8. *Under the assumptions of Proposition 2.7, we have*

$$\frac{1}{n} \sum_{k=1}^n \tau_k \xrightarrow{\mathbb{P}} \mathbb{E}_\pi(\tau_1), \quad \text{as } n \rightarrow \infty, \quad (18)$$

where π is the stationary measure of the Markov chain $\{Z_n\}_{n \geq 0}$.

Proof. Let $\varphi_n(u) := \mathbb{E}_\pi \left(e^{-un^{-1} \sum_{k=1}^n \tau_k} \right)$ and $\mathcal{F}_n := \sigma((\tilde{x}_k, \zeta_k) : k \leq n)$. By the conditional independence in (8),

$$\begin{aligned} \varphi_n(u) &= \mathbb{E} \left(\prod_{k=1}^n \mathbb{E} \left(e^{-un^{-1} \tau_k} \middle| \mathcal{F}_n \right) \right) \\ &= \mathbb{E} \left(\prod_{k=1}^n \mathbb{E} \left(e^{-un^{-1} \tau_k} \middle| \tilde{x}_{k-1}, \zeta_{k-1}, \tilde{x}_k, \zeta_k \right) \right) = \mathbb{E} \left(e^{\sum_{k=1}^n \ln G \left(\frac{u}{n} \middle| \tilde{x}_{k-1}, \zeta_{k-1}, \tilde{x}_k, \zeta_k \right)} \right). \end{aligned}$$

Since for any positive x , $\ln(x) \leq x - 1$,

$$\begin{aligned} \ln G \left(\frac{u}{n} \middle| \tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1 \right) &\leq G \left(\frac{u}{n} \middle| \tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1 \right) - 1 \\ &= \mathbb{E} \left(e^{-\frac{u}{n} \tau_1} - 1 \middle| \tilde{x}_0, \zeta_0, \tilde{x}_1, \zeta_1 \right) \leq 0, \end{aligned}$$

and, therefore, for every n ,

$$\exp \left\{ \sum_{k=1}^n \ln G \left(\frac{u}{n} \middle| \tilde{x}_{k-1}, \zeta_{k-1}, \tilde{x}_k, \zeta_k \right) \right\} \leq 1$$

and, by Dominated Convergence Theorem and Proposition 2.7, we get

$$\lim_{n \rightarrow \infty} \varphi_n(u) = \lim_{n \rightarrow \infty} \mathbb{E} \left(e^{-\frac{u}{n} \sum_{k=1}^n \tau_k} \right) = e^{-u \mathbb{E}_\pi(\tau_1)}.$$

Finally, since τ_k is supported on the positive numbers, by the continuity theorem for Laplace transforms (see theorem 2 in section XIII.1 in [8]), we obtain (18). $\square$

2.3. Long-run dynamics of the price process

In this section, we obtain a diffusive approximation for the dynamics of the midprice process of the model defined in Section 2.1. Throughout, s_t denotes the stock's midprice at time $t \in [0, \infty)$, while τ_n represents the time elapsed between the $(n-1)$ th and the n th price change as described in Section 2.1. Let $\{\tilde{u}_n\}_{n \geq 1}$ be the sequence of midprice changes. Clearly, our assumptions for the LOB dynamics described in Section 2.1 imply that $\tilde{u}_n \in \{-1/2, 1/2\}$. It is also easy to see that the midprice process is given by

$$s_t := s_0 + \sum_{j=1}^{N_t} \tilde{u}_j, \tag{19}$$

where hereafter $N_t := \max\{n \mid \tau_1 + \dots + \tau_n \leq t\}$ denotes the number of price changes up to time t . In this section, we establish the relation (1), for some constants $\sigma > 0$ and m .

Recall from Section 2.1 that $U_n := (\tilde{x}_n, \zeta_n) = ((x_n^b, x_n^a), \zeta_n)$, the number of orders in the level I of the book and the spread after the n th price change, is a Markov chain (cf. Eqs. (6)–(7)). Also, recall that, for $i \geq 0$, $Q^{a,i}(x)$ and $Q^{b,i}(x)$ are independent continuous-time Markov processes with common generator defined by (1). Define $\varsigma_n := \inf\{t > 0 : Q_t^{a,n}(\tilde{x}_{n-1}^a) \wedge Q_t^{b,n}(\tilde{x}_{n-1}^b) = 0\}$

and also consider the following events:

$$\begin{aligned} A_n &= \left\{ Q_{\varsigma_n}^{b,n}(\tilde{x}_{n-1}^b) = 0, \zeta_{n-1} = 1 \text{ or } Q_{\varsigma_n}^{b,n}(\tilde{x}_{n-1}^b) = 0, \zeta_{n-1} > 1, L_n(\zeta_{n-1}) \geq \varsigma_n \right\}, \\ B_n &= \left\{ L_n(\zeta_{n-1}) < \varsigma_n, L_n(\zeta_{n-1}) = L_n^a(\zeta_{n-1}), \zeta_{n-1} > 1 \right\}, \\ C_n &= \left\{ Q_{\varsigma_n}^{a,n}(\tilde{x}_{n-1}^a) = 0, \zeta_{n-1} = 1 \text{ or } L_n(\zeta_{n-1}) \geq \varsigma_n, Q_{\varsigma_n}^{a,n}(\tilde{x}_{n-1}^a) = 0, \zeta_{n-1} > 1 \right\}, \\ D_n &= \left\{ L_n(\zeta_{n-1}) < \varsigma_n, L_n(\zeta_{n-1}) = L_n^b(\zeta_{n-1}), \zeta_{n-1} > 1 \right\}, \end{aligned}$$

where $\{L_k^a(\zeta)\}_{k,\zeta \in \mathbb{Z}_+}$, $\{L_k^b(\zeta)\}_{k,\zeta \in \mathbb{Z}_+}$ and $\{L_k(\zeta)\}_{k,\zeta \in \mathbb{Z}_+}$ are the random variables defined in Section 2.1.

A positive price change would occur at time T_n if, either the ask queue got depleted (event A_n above) or a new queue arrived at the bid side (event B_n). Similarly, a negative price change would occur if either the bid queue got depleted (event C_n) or a new queue arrived at the ask side (event D_n). Therefore,

$$\tilde{u}_n := \frac{1}{2} [\mathbb{1}_{\{A_n\}} + \mathbb{1}_{\{B_n\}}] - \frac{1}{2} [\mathbb{1}_{\{C_n\}} + \mathbb{1}_{\{D_n\}}], \quad (20)$$

represents the n th price change, for $n \geq 1$.

As in the preceding section, an important step for analyzing the price changes would be to express those in terms of an appropriate Markov chain. Let $\Lambda := \{\bar{z} = (y_1, c_1, u) : y_1 \in \Omega_{N^*}^2, c_1 \in \mathbb{Z}_+, u \in \{-1/2, 1/2\}\}$ and

$$V_n := (\tilde{x}_n, \zeta_n, \tilde{u}_n), \quad (21)$$

for $n \geq 1$. Note that $V := \{V_n\}_{n \geq 0}$ is a Markov chain over Λ since $\tilde{x}_n, \zeta_n$ and $\tilde{u}_n$ depend only on $(\tilde{x}_{n-1}, \zeta_{n-1})$. Moreover, one can see that the states of V communicate to one another and, thus, V is irreducible. Also, provided that the assumptions of Lemmas 2.4 and 2.5 hold, one can prove that V is recurrent, similarly to the proof of Theorem 2.3, and V will then be Harris recurrent due to the countability of V 's state space. As a consequence, V would also be positive Harris. Hereafter, we denote the stationary measure and the transition probabilities of V by ν and $P^{ext}(\bar{y}, \bar{z})$, respectively.

As mentioned above, our main goal is to establish the coarse-grained behavior of the price process (19). In order to do so, we first analyze the convergence of the process $W^n := \sum_{j=1}^n \tilde{u}_j$, properly rescaled. To this end, the following Functional Central Limit Theorem (FCLT) for Markov Chains on a countable state space will be useful:

Theorem 2.9 (Meyn and Tweedie [14], Theorem 17.4.4). *Suppose that $\{V_n\}_{n \geq 0}$ is positive Harris on a countable state space Λ with transition and stationary probability measures P^{ext} and ν , respectively. Let h be a function on Λ for which a solution $\hat{h}$ to the Poisson equation,*

$$\hat{h} - P^{ext} \hat{h} = h - \nu(h), \quad (22)$$

exists with $\nu(\hat{h}^2) < \infty$. Consider the partial sums of the centered functional $\bar{h}(V_k) := h(V_k) - \nu(h)$,

$$S_n(\bar{h}) := \sum_{k=1}^n \bar{h}(V_k), \quad (23)$$

and let $r_n(t)$ be the continuous piece-wise linear function that interpolates the values of $\{S_n(\bar{h})\}_{n \geq 0}$; i.e.,

$$r_n(t) := S_{[nt]}(\bar{h}) + (nt - [nt]) [S_{[nt]+1}(\bar{h}) - S_{[nt]}(\bar{h})]. \quad (24)$$

Then, if the constant

$$\gamma^2(h) := v \left(\hat{h}^2 - (P^{ext} \hat{h})^2 \right) \quad (25)$$

is positive, it holds that,

$$\left\{ \frac{r_n(t)}{\sqrt{n\gamma^2(h)}} \right\}_{t \geq 0} \Rightarrow \{W_t\}_{t \geq 0}, \quad n \rightarrow \infty. \quad (26)$$

Remark 2.10. By taking $t = 1$, it follows that

$$\sqrt{n} \left(\frac{1}{n} \sum_{k=1}^n h(V_k) - v(h) \right) \Rightarrow \mathcal{N}(0, \gamma^2(h)).$$

The proof of the next result is deferred to the [Appendix](#).

Theorem 2.11. Let $V := \{V_n\}_{n \geq 1} = \{(\tilde{x}_n, \zeta_n, \tilde{u}_n)\}_{n \geq 1}$ be the Markov chain defined on (21) with stationary probability measure v . Then, for $h : \Lambda \rightarrow \mathbb{R}$ given by $h(x, c, u) = u$, there exists a solution to the Poisson equation $\bar{h}$ with $v(\bar{h}^2) < \infty$. Furthermore, the invariance principle (26) holds true and the variance $\gamma^2(h)$ admits the representation

$$\gamma^2(h) = \mathbb{E}_v \left(\bar{h}^2(V_1) \right) + 2 \sum_{k=2}^{\infty} \mathbb{E}_v \left(\bar{h}(V_1) \bar{h}(V_k) \right), \quad (27)$$

where $\bar{h} = h - v(h)$ and the sum converges absolutely.

In the following, we will write $f_n \stackrel{\mathbb{P}}{\sim} g_n$ if $\lim_{n \rightarrow \infty} f_n/g_n = 1$, in probability. The following result is the final ingredient towards (1):

Lemma 2.12. Using the notation of Section 2.2,

$$N_{t_n} \stackrel{\mathbb{P}}{\sim} \frac{tn}{\mathbb{E}_{\pi}(\tau_1)}, \quad \text{as } n \rightarrow \infty, \quad (28)$$

where we recall that $N_t = \max\{n \mid \tau_1 + \dots + \tau_n \leq t\}$ and π is the stationary measure of the chain $Z_n = (\tilde{x}_{n-1}, \zeta_{n-1}, \tilde{x}_n, \zeta_n)$, whose existence is guaranteed by [Theorem 2.3](#).

Proof. Throughout, let $t_n := tn$. Since N_{t_n} denote the number of price changes up to time t_n ,

$$\frac{\tau_1 + \dots + \tau_{N_{t_n}}}{N_{t_n}} \leq \frac{t_n}{N_{t_n}} < \frac{\tau_1 + \dots + \tau_{N_{t_n}+1}}{N_{t_n}}$$

and, thus, by (18), as $n \rightarrow \infty$, $\frac{t_n}{N_{t_n}} \xrightarrow{\mathbb{P}} \mathbb{E}_{\pi}(\tau_1)$, which in turn implies (28). $\square$

Finally, we can state the main result on this section.

Theorem 2.13. Let $\{s_t\}_{t \geq 0}$ be the price process as defined in Eq. (19). Then,

$$\left\{ \sqrt{n} \left(\frac{s_{tn}}{n} - \frac{v(h)}{\mathbb{E}_\pi(\tau_1)} t \right) \right\}_{t \geq 0} \Rightarrow \{\gamma(h) W_t\}_{t \geq 0}, \quad \text{as } n \rightarrow \infty, \quad (29)$$

where the variance $\gamma^2(h)$ is given as in Eq. (27).

Proof. Throughout, let $t_n := tn$. Let us recall that $s_t = s_0 + \sum_{j=1}^{N_t} \tilde{u}_j$ and $\tilde{u}_n = h(V_n)$, for the Markov chain $\{V_n\}_{n \geq 0}$ and $h: \Lambda \rightarrow \mathbb{R}$ given by $h(y, c, u) = u$. Now, we decompose the process $\bar{s}_{t_n} := n^{1/2} (s_{t_n}/n - tv(h)/\mathbb{E}_\pi(\tau_1))$ as:

$$\begin{aligned} \bar{s}_{t_n} = & \underbrace{\frac{s_0}{\sqrt{n}}}_{I_n} + \underbrace{\frac{1}{\sqrt{n}} \sum_{j=1}^{[tn/\mathbb{E}_\pi(\tau_1)]} (\tilde{u}_j - v(h))}_{II_n} + \underbrace{\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{N_{t_n}} \tilde{u}_j - \frac{1}{\sqrt{n}} \sum_{j=1}^{[tn/\mathbb{E}_\pi(\tau_1)]} \tilde{u}_j \right)}_{III_n} \\ & + \underbrace{\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{[tn/\mathbb{E}_\pi(\tau_1)]} v(h) - \sqrt{n} \frac{tv(h)}{\mathbb{E}_\pi(\tau_1)} \right)}_{IV_n}, \end{aligned}$$

where, as in Theorem 2.11, v is the stationary measure of the Markov chain $\{V_n\}_{n \geq 0}$. As $n \rightarrow \infty$, clearly, $I_n \Rightarrow 0$. Also, by Theorem 2.11,

$$II_n \Rightarrow \gamma(h)^2 W_t,$$

where $\gamma^2(h)$ is given by Eq. (27). Now, since $\tilde{u}_j \in \left\{ \frac{1}{2}, -\frac{1}{2} \right\}$, for any $\epsilon > 0$,

$$\begin{aligned} \mathbb{P} \left(\left| \sum_{j=1}^{N_{t_n}} \tilde{u}_j - \sum_{j=1}^{[tn/\mathbb{E}_\pi(\tau_1)]} \tilde{u}_j \right| \geq \epsilon \sqrt{n} \right) & \leq \mathbb{P} \left(\left| \sum_{j=N_{t_n} \wedge [tn/\mathbb{E}_\pi(\tau_1)]}^{N_{t_n} \vee [tn/\mathbb{E}_\pi(\tau_1)]} \tilde{u}_j \right| \geq \epsilon \sqrt{n} \right) \\ & \leq \mathbb{P} \left(\frac{1}{2} |N_{t_n} - [tn/\mathbb{E}_\pi(\tau_1)]| \geq \epsilon \sqrt{n} \right) \\ & \leq \mathbb{P} \left(\left| \frac{N_{t_n}}{[tn/\mathbb{E}_\pi(\tau_1)]} - 1 \right| \geq \frac{2\epsilon \sqrt{n}}{[tn/\mathbb{E}_\pi(\tau_1)]} \right), \end{aligned}$$

which, by Lemma 2.12, converges to 0 as $n \rightarrow \infty$. Thus, III_n converges to 0 in probability. Finally, since $IV_n = v(h) \frac{[tn/\mathbb{E}_\pi(\tau_1)]}{\sqrt{n}} - \sqrt{n} \frac{tv(h)}{\mathbb{E}_\pi(\tau_1)}$ is such that $0 \leq -IV_n < \frac{v(h)}{\sqrt{n}}$, it follows that $IV_n \rightarrow 0$, as $n \rightarrow \infty$, and, thus, we conclude (29).

3. Computation of some LOB features of interest

In this section we develop some numerical tools to evaluate some LOB model features of practical relevance such as the distribution of the time span between price changes, the probability of a price increase, and the probability of two consecutive price increments. The proposed method is based on an explicit characterization of the joint distribution of the time and position at which a certain two-dimensional Markov chain starting in the first quadrant hits the coordinate axes. The developed tools will also be used in Section 4 to devise an efficient simulation algorithm for the midprice dynamics of the order book.

Recall that $\bar{\Omega}_{N^*} := \{0, 1, 2, \dots, N^*\}$ and $\Omega_{N^*} := \{1, 2, \dots, N^*\}$. Throughout this section, we let $\{Y(x, y)\}_{(x,y) \in \Omega_{N^*}^2}$ be a collection of independent processes such that, for each $(x, y) \in \Omega_{N^*}^2$,

$$Y(x, y) := \{Y_t(x, y)\}_{t \in \mathbb{N}} := \left\{ \left(Q_t^{a,0}(x), Q_t^{b,0}(y) \right) \right\}_{t \in \mathbb{N}}, \quad (30)$$

where $Q^{a,0}(x) := \{Q_t^{a,0}(x)\}_{t \geq 0}$ and $Q^{b,0}(y) := \{Q_t^{b,0}(y)\}_{t \geq 0}$ are defined as in Section 2.1 (see Eq. (3)). We also set

$$\begin{aligned} \mathcal{A}_A &:= \{(0, 1), (0, 2), \dots, (0, N^*)\}, & \mathcal{A}_B &:= \{(1, 0), (2, 0), \dots, (N^*, 0)\}, \\ \mathcal{A} &:= \mathcal{A}_A \cup \mathcal{A}_B, \\ \varsigma(x, y) &:= \inf\{t > 0 : Y_t(x, y) \in \mathcal{A}\}, & L &:= L^a \wedge L^b, \end{aligned}$$

where L^a and L^b are independent exponential variables with parameter α . These variables are meant to represent the times for a new set of orders to arrive at the ask and bid side, respectively. Finally, $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution.

3.1. Distribution of the duration between price changes

Here, we develop a numerical method to find the distribution of the first price change time τ_1 given that, initially at time 0, there are x orders at the bid, y at the ask, and the spread is z . To this end, we first compute the joint distribution of the vector $(\varsigma(x, y), Y_{\varsigma(x,y)}(x, y))$. This is obtained via the following two lemmas, whose proofs can be found in the [Appendix](#)

Lemma 3.1. Suppose that, for each fixed $\bar{a} := (\bar{a}_1, \bar{a}_2) \in \mathcal{A}$, $u_{\bar{a}} : [0, T] \times \bar{\Omega}_{N^*}^2 \rightarrow \mathbb{R}$ satisfies the following system of differential equations:

$$\begin{aligned} \left(-\frac{\partial}{\partial t} + \mathcal{L} \right) u_{\bar{a}}(t, x, y) \Big|_{t=T-r} &= 0, \quad \text{for } 0 \leq r < T, \quad (x, y) \in \Omega_{N^*}^2 \\ u_{\bar{a}}(T-r, x, y) &= \mathbb{1}_{\{(x,y)=\bar{a}\}}, & \text{for } 0 \leq r \leq T, \quad (x, y) \in \mathcal{A}, \\ u_{\bar{a}}(0, x, y) &= \mathbb{1}_{\{(x,y)=\bar{a}\}}, & \text{for } (x, y) \in \bar{\Omega}_{N^*}^2, \end{aligned} \quad (31)$$

where $\mathcal{L}u(t, x, y)$ is the finite difference operator given by

$$\begin{aligned} \mathcal{L}u(t, x, y) &= \begin{cases} \lambda(u_1^+ + u_2^+) + v(u_1^- + u_2^-) - 2(\lambda + v)u, & (x, y) \in \{1, 2, \dots, N^* - 1\}^2, \\ \lambda u_2^+ + v(u_1^- + u_2^-) - (\lambda + 2v)u, & x = N^*, y \in \{1, 2, \dots, N^* - 1\}, \\ \lambda u_1^+ + v(u_1^- + u_2^-) - (\lambda + 2v)u, & x \in \{1, 2, \dots, N^* - 1\}, y = N^*, \\ v(u_1^- + u_2^-) - 2vu, & (x, y) = (N^*, N^*), \\ 0, & (x, y) \in \mathcal{A}, \end{cases} \end{aligned} \quad (32)$$

and $u_1^+ = u(t, x+1, y)$, $u_2^+ = u(t, x, y+1)$, $u_1^- = u(t, x-1, y)$, $u_2^- = u(t, x, y-1)$, and $u = u(t, x, y)$. Then, for $t > 0$, $(x, y) \in \bar{\Omega}_{N^*}^2$, and $\bar{a} := (\bar{a}_1, \bar{a}_2) \in \mathcal{A}$,

$$u_{\bar{a}}(t, x, y) := \mathbb{P}[\varsigma(x, y) \leq t, Y_{\varsigma(x,y)}(x, y) = \bar{a}]. \quad (33)$$

The next result proves the existence of a solution u to the system (31) by giving an explicit representation of u in terms of the eigenvalues and eigenvectors of a certain finite difference

operator. As a result, we obtain as well an explicit formulation of the joint distribution of $(\zeta(x, y), Y_{\zeta(x, y)}(x, y))$. Below, we let

$$\overline{a+1} := \begin{cases} \bar{a} + (0, 1), & \text{if } \bar{a} \in \{(1, 0), (2, 0), \dots, (N^*, 0)\} \\ \bar{a} + (1, 0), & \text{if } \bar{a} \in \{(0, 1), (0, 2), \dots, (0, N^*)\}. \end{cases}$$

Proposition 3.2. Let Δ be the symmetric finite difference operator defined for functions $w : \bar{\Omega}_{N^*}^2 \rightarrow \mathbb{R}$ as

$$\begin{aligned} \Delta w(x, y) &= \begin{cases} w_1^+ + w_2^+ + w_1^- + w_2^- - 4w, & (x, y) \in \{1, 2, \dots, N^* - 1\}^2 \\ w_2^+ + w_1^- + w_2^- - \left(4 - \sqrt{\frac{\lambda}{\nu}}\right)w, & x = N^*, y \in \{1, 2, \dots, N^* - 1\} \\ w_1^+ + w_1^- + w_2^- - \left(4 - \sqrt{\frac{\lambda}{\nu}}\right)w, & x \in \{1, 2, \dots, N^* - 1\}, y = N^* \\ w_1^- + w_2^- - \left(4 - 2\sqrt{\frac{\lambda}{\nu}}\right)w, & (x, y) = (N^*, N^*) \\ 0, & (x, y) \in \mathcal{A}, \end{cases} \end{aligned} \quad (34)$$

where $w_1^+ = w(x+1, y)$, $w_2^+ = w(x, y+1)$, $w_1^- = w(x-1, y)$, $w_2^- = w(x, y-1)$, and $w = w(x, y)$. Let $\{\xi_k\}_{k=1}^{N^*2}$ be the eigenvalues of Δ and $\{f_k(x, y)\}_{k=1}^{N^*2}$ be their corresponding eigenvectors so that they constitute an orthonormal basis of $\mathbb{R}^{N^*2}$. For $\bar{a} := (\bar{a}_1, \bar{a}_2) \in \mathcal{A}$, let $u_{\bar{a}} : [0, T] \times \bar{\Omega}_{N^*}^2 \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned} u_{\bar{a}}(t, x, y) &= \left(\frac{\lambda}{\nu}\right)^{\frac{\bar{a}_1 + \bar{a}_2 - x - y}{2}} \left[\sum_{k=1}^{N^*2} \frac{\sqrt{\lambda\nu} f_k(\overline{a+1})}{2(\lambda + \nu) - \sqrt{\lambda\nu}(4 + \xi_k)} \left(1 - e^{-t[2(\lambda + \nu) - (4 + \xi_k)\sqrt{\lambda\nu}]}\right) \right. \\ &\quad \left. \times f_k(x, y) \mathbb{1}_{\{(x, y) \in \Omega_{N^*}^2\}} + \mathbb{1}_{\{(x, y) = \bar{a}\}} \right]. \end{aligned} \quad (35)$$

Then, the function $u_{\bar{a}}$ satisfies the system of differential equations (31) and, therefore, the identity (33) holds true.

Remark 3.3. We can rewrite Eq. (35) as:

$$\begin{aligned} u_{\bar{a}}(t, x, y) &:= \chi^{\frac{\bar{a}_1 + \bar{a}_2 - x - y}{2}} \sum_{k=1}^{N^*2} \frac{f_k(\overline{a+1})}{2(\chi^{1/2} - \chi^{-1/2})^2 - \xi_k} \left(1 - e^{-2\lambda t[(\chi^{-1/2} - 1)^2 - \frac{\xi_k}{2}\chi^{-1/2}]}\right) \\ &\quad \times f_k(x, y) \mathbb{1}_{\{(x, y) \in \Omega_{N^*}^2\}} + \chi^{\frac{\bar{a}_1 + \bar{a}_2 - x - y}{2}} \mathbb{1}_{\{(x, y) = \bar{a}\}}, \end{aligned}$$

where $\chi := \lambda/\nu$. The previous expression shows that, as t gets larger, the joint probability distribution $P[\zeta(x, y) \leq t, Y_{\zeta(x, y)}(x, y) = \bar{a}]$ depends on the parameters ν and λ mostly through the quotient $\chi = \lambda/\nu$. Let us also point out that the eigenvalues of Δ can be proven to be non-positive and, thus, $u_{\bar{a}}(T, x, y) \in [0, 1]$.

We are now ready to compute the distribution $F_{\tau_1}(t|x, y, z) := \mathbb{P}[\tau_1 \leq t | x_0^a = x, x_0^b = y, \zeta_0 = z]$ of the time τ_1 it takes for the price to change conditioned on the initial state of the book. For simplicity of notation, throughout $\tau(x, y, z)$ represents a random time such that $\mathbb{P}(\tau(x, y, z) \leq t) = \mathbb{P}[\tau_1 \leq t | x_0^a = x, x_0^b = y, \zeta_0 = z]$, for any $t \geq 0$. It is clear that $\tau(x, y, 1) \stackrel{\mathcal{D}}{=} \varsigma(x, y)$ and, thus, from Eq. (35), for $(x, y) \notin \mathcal{A}$,

$$F_{\tau_1}(t|x, y, z) = \left(\frac{\lambda}{\nu}\right)^{-\frac{x+y}{2}} \sum_{k=1}^{N^*2} \frac{\sqrt{\lambda\nu} \varsigma_k}{2(\lambda + \nu) - \sqrt{\lambda\nu}(4 + \xi_k)} \times \left(1 - e^{-t(2(\lambda+\nu) - (4+\xi_k)\sqrt{\lambda\nu})}\right) f_k(x, y), \quad (36)$$

where

$$\varsigma_k := \sum_{\bar{a} \in \mathcal{A}} \left(\frac{\lambda}{\nu}\right)^{\frac{\bar{a}_1 + \bar{a}_2}{2}} f_k(\bar{a} + 1). \quad (37)$$

On the other hand, for $z \geq 2$, we have that $\tau(x, y, z) \stackrel{\mathcal{D}}{=} \varsigma(x, y) \wedge L$, where as before L represents the arrival time of a limit order within the spread. Therefore, from the independence of $\varsigma(x, y)$ and L , for any $z \geq 2$ and $(x, y) \notin \mathcal{A}$,

$$F_{\tau_1}(t|x, y, z) = \mathbb{P}[L \leq t] + \mathbb{P}[\varsigma(x, y) \leq t] \mathbb{P}[L > t] = (1 - e^{-2\alpha t}) + F_{\tau_1}(t|x, y, 1)e^{-2\alpha t}. \quad (38)$$

The expressions (36)–(38) provide an efficient numerical method to compute the distribution of the time span between price changes given some initial level I LOB setup. The method is relatively efficient since the main task in their evaluation is the computation of the eigenvalues $\{\xi_k\}_{k=1}^{N^*2}$ and eigenvectors $\{f_k(x, y)\}_{k=1}^{N^*2}$, which has to be done only once, for any $t \geq 0$ and $z \in \{1, 2, \dots\}$.

3.2. Probability of a price increase

We now consider the probability of a price increase conditioned on the current state of the order book:

$$p(x, y, z) := \mathbb{P}[\text{Price increase} | x \text{ orders at Bid, } y \text{ orders at Ask, and a spread } z], \\ \text{for } (x, y) \notin \mathcal{A}.$$

A price increase occurs if the best ask queue gets depleted or if a new set of orders arrives at the bid side. Recall from Lemma 3.1 that $u_{\bar{a}}(t, x, y) := \mathbb{P}[\varsigma(x, y) \leq t, Y_{\varsigma(x, y)}(x, y) = \bar{a}]$ has an explicit form given by Eq. (35). Set

$$u_B(t, x, y) := \mathbb{P}[\varsigma(x, y) \leq t, Y_{\varsigma(x, y)} \in \mathcal{A}_B] = \sum_{\bar{a} \in \mathcal{A}_B} u_{\bar{a}}(t, x, y), \quad (39)$$

and note that, if the spread is $z = 1$,

$$p(x, y, 1) = u_B(\infty, x, y) = \left(\frac{\lambda}{\nu}\right)^{-\frac{x+y}{2}} \sum_{k=1}^{N^*2} \frac{\sqrt{\lambda\nu} \varsigma_{k,B}}{2(\lambda + \nu) - \sqrt{\lambda\nu}(4 + \xi_k)} f_k(x, y), \quad (40)$$

where

$$\varsigma_{k,B} := \sum_{\bar{a} \in \mathcal{A}_B} \left(\frac{\lambda}{v} \right)^{\frac{\bar{a}_1 + \bar{a}_2}{2}} f_k(\bar{a} + 1).$$

In order to find $p(x, y, z)$ for $z \geq 2$, note that

$$\begin{aligned} p(x, y, z) &= \mathbb{P}[\zeta(x, y) \leq L, Y_{\zeta(x, y)}(x, y) \in \mathcal{A}_B] + \mathbb{P}[\zeta(x, y) > L, L = L^b] \\ &=: p_1(x, y) + p_2(x, y). \end{aligned} \quad (41)$$

By conditioning on L and recalling that $L \sim \exp(2\alpha)$,

$$\begin{aligned} p_1(x, y) &= 2\alpha \int_0^\infty u_B(t, x, y) e^{-2\alpha t} dt \\ &= \left(\frac{\lambda}{v} \right)^{-\frac{x+y}{2}} \sum_{k=1}^{N^*2} \frac{\sqrt{\lambda v} \varsigma_{k,B}}{2(\lambda + v) - \sqrt{\lambda v}(4 + \xi_k)} \\ &\quad \times \left(1 - \frac{2\alpha}{2(\lambda + v + \alpha) - (4 + \xi_k)\sqrt{\lambda v}} \right) f_k(x, y). \end{aligned} \quad (42)$$

For the second term, using the symmetry between L^a and L^b , $p_2(x, y) = \frac{1}{2} \mathbb{P}[\zeta(x, y) \geq N]$ and, thus,

$$\begin{aligned} p_2(x, y) &= \frac{1}{2} \left(1 - 2\alpha \int_0^\infty \mathbb{P}[\zeta(x, y) \leq t] e^{-2\alpha t} dt \right) \\ &= \frac{1}{2} \left(1 - \left(\frac{\lambda}{v} \right)^{-\frac{x+y}{2}} \sum_{k=1}^{N^*2} \frac{\sqrt{\lambda v} \varsigma_k}{2(\lambda + v) - \sqrt{\lambda v}(4 + \xi_k)} \right. \\ &\quad \times \left. \left(1 - \frac{2\alpha}{2(\lambda + v + \alpha) - (4 + \xi_k)\sqrt{\lambda v}} \right) f_k(x, y) \right), \end{aligned} \quad (43)$$

where ς_k is defined as in (37). Again, once the eigenvalues and eigenvectors of Δ have been computed, one can readily compute $p(x, y, z)$ via (42)–(43), for any $(x, y) \in \Omega_{N^*}^2$ and $z \in \mathbb{Z}_+$.

3.3. Probability of two consecutive price increments

Let $\hat{p}(x, y, z)$ be the probability of two consecutive increments in the price given that initially there were x orders at the best bid, y orders at the best ask, and a spread of z . These probabilities are highly dependent on the initial spread. The case of an initial spread of 1 is relatively easier to analyze than any other spread due to the possibility of a new set of orders within the spread before the depletion of any of the level I queues. As will be shown below, in the latter situation, we will have to consider a probability of the form $\mathbb{P}[L < \zeta(x, y), Y_L(x, y) \in \{(1, j), \dots, (N^*, j)\}]$, for any j . The aforementioned probability will be reformulated in terms of the solution to a certain initial value problem along the lines of Proposition 3.2.

Recall that every time there is a price change, a new number of orders in the LOB side that got depleted is generated from a discrete distribution, f^a or f^b , supported on $\{1, 2, \dots, N^*\}$, depending on whether the best ask or bid queues got depleted. For simplicity, in what follows

we assume that $f := f^a = f^b$. Denote H a random variable with distribution f . In addition to the collection of random walks $\{Y(x, y)\}_{(x,y) \in \Omega_{N^*}^2}$ described at the beginning of Section 3.1, we also need to consider another independent copy $\{\tilde{Y}(x, y)\}_{(x,y) \in \Omega_{N^*}^2}$ and fix $\tilde{\zeta}(x, y) := \inf\{t > 0 : \tilde{Y}_t(x, y) \in \mathcal{A}\}$. Similarly, in addition to (L^a, L^b) , we consider an independent copy $(\tilde{L}^a, \tilde{L}^b)$ and fix $\tilde{L} := \tilde{L}^a \wedge \tilde{L}^b$. We are ready to compute $\hat{p}(x, y, z)$.

For $z = 1$, clearly,

$$\begin{aligned} \hat{p}(x, y, 1) &= \sum_{i=1}^{N^*} \sum_{j=1}^{N^*} \mathbb{P} \left[Y_{\zeta(x,y)}(x, y) = (j, 0), H = i, \tilde{\zeta}(j, i) \leq \tilde{L}, \tilde{Y}_{\tilde{\zeta}(j,i)}^2(j, i) \in \mathcal{A}_B \right] \\ &\quad + \sum_{i=1}^{N^*} \sum_{j=1}^{N^*} \mathbb{P} \left[Y_{\zeta(x,y)}(x, y) = (j, 0), H = i, \tilde{\zeta}(j, i) \geq \tilde{L}, \tilde{L} = \tilde{L}^b \right] \\ &= \sum_{i=1}^{N^*} \sum_{j=1}^{N^*} u_{(j,0)}(\infty, x, y) f(i) p(j, i, 2), \end{aligned}$$

where we recall that $p(x, y, 2)$ denotes the probability of a price increase if there are x orders at the bid, y orders at the ask, and a spread of 2. The probability $p(x, y, 2)$ can be computed according to (41), while $u_{(j,0)}(\infty, x, y)$ can readily be found from (35) by making $t \rightarrow \infty$. It is worth mentioning that the case $z = 1$ is arguably the most important in practice since, as empirically observed in several studies, the spread spends a great deal of time at level 1.

Next, let $\mathcal{A}_{B_j} := \{(1, j), (2, j), \dots, (N^*, j)\}$. Now, for $z = 2$,

$$\begin{aligned} \hat{p}(x, y, 2) &= \sum_{i=1}^{N^*} \sum_{j=1}^{N^*} \mathbb{P}[\zeta(x, y) \leq L, Y_{\zeta(x,y)}(x, y) = (j, 0), H = i, \tilde{\zeta}(j, i) \leq \tilde{L}, \\ &\quad \tilde{Y}_{\tilde{\zeta}(j,i)}(j, i) \in \mathcal{A}_B] \\ &\quad + \sum_{i=1}^{N^*} \sum_{j=1}^{N^*} \mathbb{P}[\zeta(x, y) \leq L, Y_{\zeta(x,y)}(x, y) = (j, 0), H = i, \tilde{\zeta}(j, i) \geq \tilde{L}, \tilde{L} = \tilde{L}^b] \\ &\quad + \sum_{i=1}^{N^*} \sum_{j=1}^{N^*} \mathbb{P}[L < \zeta(x, y), L = L^b, Y_L(x, y) \in \mathcal{A}_{B_j}, H = i, \tilde{Y}_{\tilde{\zeta}(i,j)}(i, j) \in \mathcal{A}_B]. \end{aligned}$$

Hence, using that $\mathbb{P}[L \leq \zeta(x, y), L = L^a, Y_a(x, y) \in \mathcal{A}_{B_j}] = \mathbb{P}[L \leq \zeta(x, y), L = L^b, Y_L(x, y) \in \mathcal{A}_{B_j}]$, we can write

$$\begin{aligned} \hat{p}(x, y, 2) &= \sum_{i=1}^{N^*} \sum_{j=1}^{N^*} f(i) \left\{ \left(2\alpha \int_0^\infty u_{(j,0)}(t, x, y) e^{-2\alpha t} dt \right) p(j, i, 2) \right. \\ &\quad \left. + \frac{1}{2} \mathbb{P}[L < \zeta(x, y), Y_L(x, y) \in \mathcal{A}_{B_j}] p(i, j, 1) \right\}. \end{aligned}$$

The probability $p(x, y, 1)$ can be computed according to (40), while $2\alpha \int_0^\infty u_{(j,0)}(t, x, y) e^{-2\alpha t} dt$ can readily be found from (35). The problem of computing $\mathbb{P}[L \leq \zeta(x, y), Y_L(x, y) \in \mathcal{A}_{B_j}]$ is

analyzed below. Before that, let us note that, using similar arguments,

$$\hat{p}(x, y, 3) = \sum_{i=1}^{N^*} \sum_{j=1}^{N^*} f(i) \left\{ \left(2\alpha \int_0^\infty u_{(j,0)}(t, x, y) e^{-2\alpha t} dt \right) p(j, i, 2) + \frac{1}{2} \mathbb{P}[L < \varsigma(x, y), Y_L(x, y) \in \mathcal{A}_{B_j}] p(i, j, 2) \right\}.$$

A similar identity holds for $\hat{p}(x, y, z)$ with $z \geq 4$. Therefore, the only remaining step is the computation of $\mathbb{P}[L \leq \varsigma(x, y), Y_L(x, y) \in \mathcal{A}_{B_j}]$. This can be done by first computing $v_j(t, x, y) := \mathbb{P}[t < \varsigma(x, y), Y_t(x, y) \in \mathcal{A}_{B_j}]$ using similar arguments to those used in Proposition 3.2. More concretely, it turns out that $v_j(t, x, y)$ solves the initial value problem:

$$\begin{cases} \left(-\frac{\partial}{\partial t} + \mathcal{L} \right) v_j(t, x, y) \Big|_{t=T-r} = 0 & \text{for } 0 \leq r \leq T, (x, y) \in \{1, 2, \dots, N^*\}^2, \\ v_j(T-r, x, y) = 0 & \text{for } 0 \leq r \leq T, (x, y) \in \mathcal{A}, \\ v_j(0, x, y) = \mathbb{1}_{\{(x,y) \in \mathcal{A}_{B_j}\}} & \text{for } (x, y) \in \{0, 1, 2, \dots, N^*\}^2. \end{cases} \quad (44)$$

4. Numerical examples

The purpose of this section is twofold. First, we analyze numerically the convergence of the midprice process towards its limiting diffusive process as established by Theorem 2.13. Second, we compute some of the quantities of interest described in Section 3 and numerically study their behaviors under our assumptions and those in [6]. For the first problem, we develop a new simulation scheme for the price process, which is more efficient than the direct simulation of all the LOB events (i.e., limit, market, and cancellation orders).

Recall that the input parameters of the model are the rates λ , μ , θ , and α . The first three parameters refer to the arrival rates of limit orders, market orders, and cancellation, respectively, while α is the rate at which a new set of limit orders arrive in-between the bid–ask spread. Also, the distributions f^b and f^a , for the sizes of queues at the best bid and ask price after the respective best bid and ask price changes, have to be considered. For simplicity, we set $f := f^a = f^b$ and recall that we are assuming that f^a, f^b are supported on the finite set $\{1, \dots, N^*\}$.

In the subsequent numerical examples, we use the empirically estimated intensities described in Table 1, which are borrowed from [6] (see Table 3 therein). All the time measurements are in seconds. The maximum queue size N^* is assumed to be 10 units, with each unit representing a batch of 100 shares. Unless otherwise specified, the initial level I queue's configuration is set to be $(x, y) = (5, 5)$, while the initial spread is $\zeta_0 = 4$. The distribution f is taken to be uniformly distributed in $\{1, \dots, N^*\}$. Finally, two different choices of α are considered: $\alpha = \nu + 1$ and $\alpha = 2\nu$.

4.1. Simulation and convergence assessment

The most natural (and naive) way to simulate the price dynamics consists of generating all the LOB events or, equivalently, all the arrival times of orders (limit, market, and cancellations) from the corresponding Poisson process, until the time that either the bid or ask queue gets depleted or a new set of limit orders arrives within the spread (if possible). We then reset the

Table 1

Estimates for the intensities of limit orders and market orders+cancellations, in number of batches per second (each batch representing 100 shares) on June 26th, 2008, as reported in [6].

Stock	λ	$\nu := \mu + \theta$
Citigroup	2204	2331
General electric	317	325
General motors	102	104

Table 2

Estimates of the expected return $\mathbb{E}[s_t]/t$, normalized variance $\text{Var}(s_t)/t$, and expected time $\mathbb{E}_\pi(\tau)$ between price changes.

Case	Scenario 1: $\lambda = 2204, \nu = 2331$				Scenario 2: $\lambda = 317, \nu = 325$				Scenario 3: $\lambda = 102, \nu = 104$			
	$\alpha = 2332$		$\alpha = 4662$		$\alpha = 326$		$\alpha = 650$		$\alpha = 105$		$\alpha = 210$	
	$t =$	$t =$	$t =$	$t =$	$t =$	$t =$	$t =$	$t =$	$t =$	$t =$	$t =$	$t =$
	60	300	60	300	60	300	60	300	60	300	60	300
$\mathbb{E}[s_t]/t$	−0.93	−0.90	−0.86	−0.96	−0.09	−0.15	−0.21	−0.23	−0.13	−0.09	−0.09	−0.06
$\text{Var}[s_t]/t$	291.51	280.64	348.89	278.84	39.08	34.48	36.62	41.87	10.68	13.68	11.92	12.93
$\mathbb{E}[t/N_t]$	$\frac{32}{1000}$	$\frac{32}{1000}$	$\frac{32}{1000}$	$\frac{32}{1000}$	$\frac{245}{1000}$	$\frac{245}{1000}$	$\frac{247}{1000}$	$\frac{247}{1000}$	$\frac{777}{1000}$	$\frac{778}{1000}$	$\frac{783}{1000}$	$\frac{784}{1000}$

queue size at the side that was changed and continue this process. Unfortunately, this procedure is computationally intensive to study the coarse-grain behavior of the price process, especially for a Monte Carlo analysis, in which we require a large number of simulations. Instead, we propose a more efficient method, in which, without simulating the events leading to it, we directly simulate the random vector $(\zeta(x, y), Y_{\zeta(x, y)}(x, y))$, which represents the time of a depletion at the level I and the amount of orders in the level at such a time. This in turn would allow us to obtain the time of a price change and the amount of outstanding limit orders at the opposite side of the book. To simulate $(\zeta(x, y), Y_{\zeta(x, y)}(x, y))$, we take advantage of the representation for their joint probability given by Eq. (35). This representation has several advantages. In particular, its computation requires to find the eigenvalues $\{\xi_k\}$ and eigenfunctions $\{f_k(x, y)\}$ of the discrete Laplacian, *only once*, regardless of t and $\bar{a}$.

By Lemma 2.12 and Theorem 2.13, we have

$$\mathbb{E}\left(\frac{s_t}{t}\right) \xrightarrow{t \rightarrow \infty} \frac{\nu(h)}{\mathbb{E}_\pi(\tau_1)}, \quad \frac{\text{Var}(s_t)}{t} \xrightarrow{t \rightarrow \infty} \gamma^2(h), \quad \mathbb{E}\left(\frac{t}{N_t}\right) \xrightarrow{t \rightarrow \infty} \mathbb{E}_\pi(\tau_1). \quad (45)$$

We proceed to analyze the performance of the above asymptotic approximations for “large” t . Our goal is to assess how close the distribution of s_t is to its diffusive approximation for some sampling time spans t commonly used in practice (say, 1, 5 min, etc.). To compute the expectations and variances appearing in (45), we use a Monte Carlo method with 200 simulations of the orderbook. The results are shown in Table 2. As expected, the larger the rates λ and ν are, the smaller $\mathbb{E}_\pi(\tau_1)$ gets and, as a result, the larger the expected rate of return $\mathbb{E}(s_t)$ becomes. We also observe that, in that case, there seems to be a significant increment in the volatility $\sqrt{\text{Var}(s_t)}$ of the asset price. An intuitive interpretation of this phenomenon is that increasing λ and ν simultaneously is equivalent to speeding up the dynamics of the process, which will necessarily result in higher variability.

Table 3
Proportion of time spent by the spread at the different values during a 5 min time period.

Case		1 Tick	2 Ticks	3 Ticks	4+ Ticks
$\alpha = \nu + 1$	$\lambda = 2204, \nu = 2331$	0.97248	0.02716	0.00035	0.00001
	$\lambda = 317, \nu = 325$	0.906891	0.088491	0.004135	0.000564
	$\lambda = 102, \nu = 104$	0.86881	0.12084	0.008868	0.001482
$\alpha = 2\nu$	$\lambda = 2204, \nu = 2331$	0.98627	0.01365	0.00007	0.00001
	$\lambda = 317, \nu = 325$	0.95327	0.045697	0.000956	0.000077
	$\lambda = 102, \nu = 104$	0.94383	0.054443	0.001579	0.000148

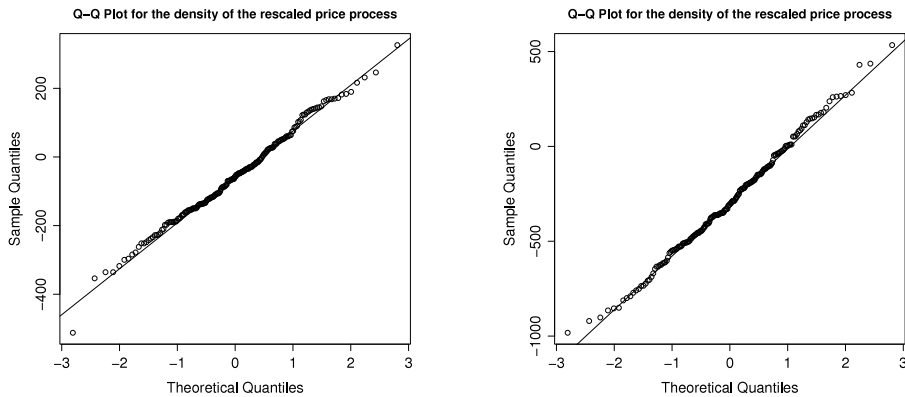


Fig. 2. QQ Normal plot for the sample of s_t , when $\lambda = 2204$, $\nu = 2331$, and $\alpha = 2332$. The time horizon chosen is $t = 60$ s (left panel) and $t = 300$ s (right panel).

Next, we turn our attention to the behavior of the spread. Based again on 200 simulations and an initial spread of 4, Table 3 shows the percentage of the time that the spread spends at each state during the time interval $[0, 300 \text{ sec}]$ for the different values of λ , ν , and α . These results show that the larger the rates λ and ν are, the longer the spread spends in the state of one tick. This is due to the fact that the larger these rates are, the quicker the spread change and, by the choice of α , the quicker it will shrink to 1. More importantly, these results show that, when α/ν is large enough, our model can closely replicate the stylized empirical behavior of the spread as suggested, for instance, by Cont and Larrard [6] (see Table 2 therein).

Finally, Figs. 2–4 compare the empirical density of s_t , based on 200 simulations, to a Gaussian distribution via QQ normal plots. We do this for $t = 1$ min and $t = 5$ min, which, for instance, are commonly used as sampling frequencies of several statistical estimation methods. For the sake of space, we only show the graphs corresponding to $\alpha = \nu + 1$ (there is no significant changes when $\alpha = 2\nu$). As seen in the graphs, the distribution of s_t is relatively well approximated by a Normal distribution for these two values of t .

4.2. Evaluation of some quantities of interest

To understand the impact of the assumptions made in our model and make some comparisons to the model presented in [6], we proceed to compute numerically some of the quantities of interest introduced in Section 3.

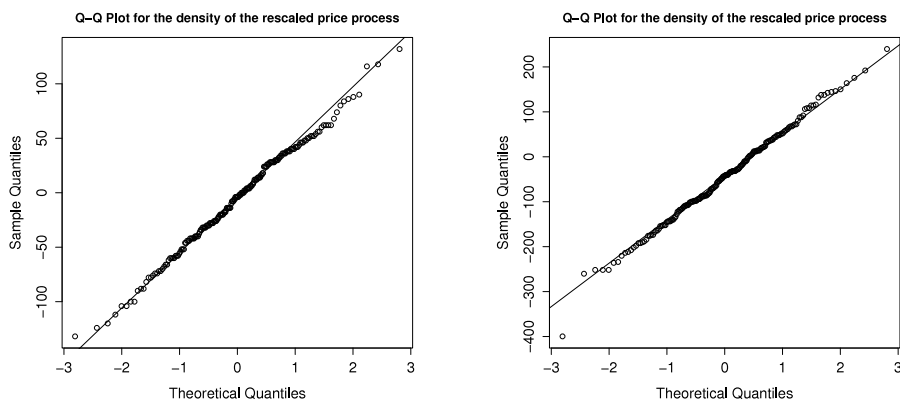


Fig. 3. QQ Normal plot for the sample of s_t , when $\lambda = 317$, $\nu = 325$, and $\alpha = 326$. The time horizon chosen is $t = 60$ s (left panel) and $t = 300$ s (right panel).

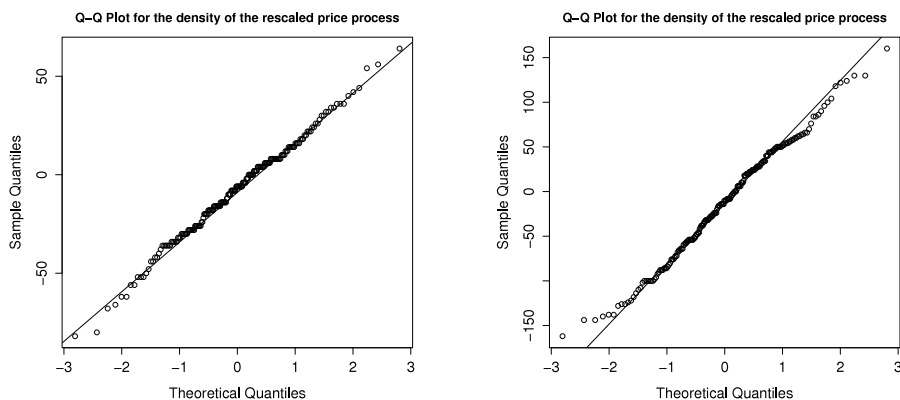


Fig. 4. QQ Normal plot for the sample of s_t , when $\lambda = 102$, $\nu = 104$, and $\alpha = 208$. The time horizon chosen is $t = 60$ s (left panel) and $t = 300$ s (right panel).

We first consider the distribution of the time span between price changes. This distribution was computed in Proposition 1 of [6] under the assumptions therein. The survival function was also plotted in Figure 4 therein with $\lambda = 12$, $\mu + \theta = 13$, $x_0^a = 5$, and $x_0^b = 4$ as the input parameters. In the left panel of Fig. 5, conditional on a spread of 1, the survival distribution of the time between price changes, τ , is reproduced and compared with the distributions obtained by the method introduced in Section 3.1 (see Eqs. (36)–(38)) for different values of N^* . The right panel of Fig. 5 also depicts the densities of the time τ . As shown in the graphs, for values of N^* close to 5, the density is more concentrated around 0, which is natural since the queue sizes cannot surpass the value of N^* , which tends to produce smaller τ values. Furthermore, as expected, the survival and density functions under our model converge to those of Cont and Larrard [6] as N^* increases.

The distribution of the time for the next price to occur, when the spread is 2, is not considered here, because this is very similar to the one of an exponential random variable with parameter 2α . This is because, under the recurrence condition $\alpha \geq \mu + \theta$, it is far more probable that a price change occurs due to the arrival of a new set of limit orders within the spread than to the depletion of a level I queue.

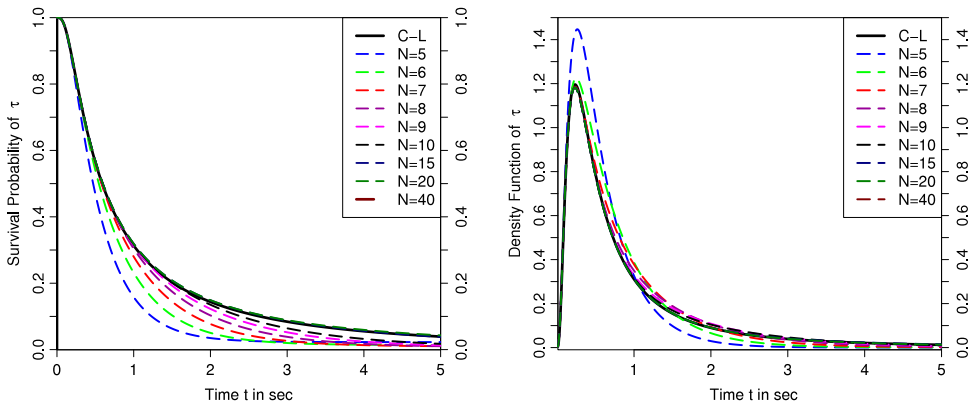


Fig. 5. Survival and density functions of the time for the first price change to occur under the Cont and Larrard's model and our model with different values of $N = N^*$, conditioning on the spread to be 1. Parameter choices: $\lambda = 12$, $\mu + \theta = 13$, $x_0^a = 5$, and $x_0^b = 4$.

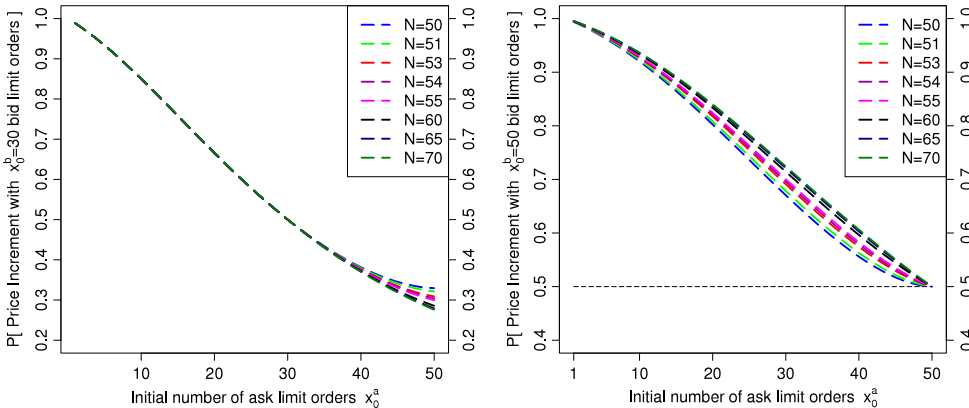


Fig. 6. Comparison of the probability of a price increase as a function of x_0^a for different values of $N = N^*$. The number of bid orders is fixed at 30 (left panel) and 50 (right panel).

Next, we compare the probability of a price increase for different values of x_0^a , x_0^b , and N^* , when the spread is set to be 1. Cont and Larrard [6] provide a formula for such a quantity in Proposition 3 therein, but, unfortunately, this formula is difficult to implement in the asymmetric order flow case. By contrast, the method proposed in Section 3.2 is more efficient, since all of the quantities therein rely on the spectral decomposition of the discrete Laplacian (34), which, once N^* is fixed, has to be computed only once. Fig. 6 shows the probability of a price increase as a function of x_0^a , for fixed $x_0^b = 30$ (left panel) and $x_0^b = 50$ (right panel). By symmetry, we would have the same graph if we plot the probability against x_0^b for fixed values of x_0^a . Note that, in the first case, when $x_0^b = 30$, the probability of a price increase does not vary significantly with N^* , for most of the values of x_0^a . It is only when x_0^a becomes close to the value of N^* that we notice some variations. On the other hand, when $x_0^b = 50$, which is closer to the considered values for N^* , the probability significantly varies with N^* , for a large range of values of x_0^a . The dashed lines therein show that, regardless of N^* , the probability of a price increase is always 0.5, as it should be, when $x_0^a = x_0^b$.

5. Conclusions

In this paper, we propose a new Markovian model for the dynamics of the level I of a limit order book. Our model incorporates a varying spread and avoids resetting completely the level I of the book at each price change, both of which are real features not accounted for in earlier works that obtain similar results to ours. Although the general rules governing the order book in this setting create a relatively complex dynamics, we are still capable of computing several features of the LOB model of relevance in high-frequency trading (see [7] for this type of applications).

Our main result characterizes the coarse-grain behavior of the midprice process in terms of a Brownian motion with drift. This was made possible by expressing the price changes in terms of a suitable Markov chain and to take advantage of the ergodic theory for countable positive recurrent Harris chains. To this end, two key assumptions were needed: the boundedness of the queue sizes at every moment and a sufficiently high arrival rate of new orders in between the spread compared to the intensity of arrivals of market order/cancellations. The latter condition is also intuitive since it prevents the spread to grow indefinitely with positive probability. These two conditions enable us to study the diffusive nature of the price process without losing realism.

It is known that markets exhibit relatively large price shifts in relatively short time periods and, thus, the incorporation of these “jumps” into an order book model is appealing. A natural approach to address this problem may be the introduction of more levels in the order book, governed by similar rules to those imposed in the model proposed in this work.

Acknowledgment

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Appendix. Additional proofs

Proof of Lemma 2.4. Throughout, we set $Y_t(x, y) := (Y_t^1(x, y), Y_t^2(x, y)) := (Q_t^{b,0}(x), Q_t^{a,0}(y))$, $L^a := L_1^a(2)$, $L^b := L_1^b(2)$, and $L := L^a \wedge L^b$. Also, let $\hat{y} = (y_1, j \pm 1, y_2, j) \in \Xi \setminus F$, where $y_1 = (y_1^1, y_1^2) \in \Omega_{N^*}^2$, $y_2 = (y_2^1, y_2^2) \in \Omega_{N^*}^2$ and $j > 1$. In that case,

$$\begin{aligned} (P\varphi)(\hat{y}) &= \sum_{z_2=(z_2^1, z_2^2) \in \Omega_{N^*}^2} P((y_1, j \pm 1, y_2, j), (y_2, j, z_2, j-1)) \varphi((y_2, j, z_2, j-1)) \\ &\quad + \sum_{z_2=(z_2^1, z_2^2) \in \Omega_{N^*}^2} P((y_1, j \pm 1, y_2, j), (y_2, j, z_2, j+1)) \varphi((y_2, j, z_2, j+1)), \end{aligned}$$

which, using that $\varphi((y_2, j, z_2, k)) = \varphi(k)$, can then be decomposed and simplified as follows:

$$\begin{aligned} (P\varphi)(\hat{y}) &= \sum_{z_2^1=1}^{N^*} \sum_{z_2^2=1}^{N^*} \mathbb{P}(L < \varsigma(y_2), L = L^a, Y_L^1(y_2) = z_2^1) f^a(z_2^2) \varphi(j-1) \\ &\quad + \sum_{z_2^1=1}^{N^*} \sum_{z_2^2=1}^{N^*} \mathbb{P}(L < \varsigma(y_2), L = L^b, Y_L^2(y_2) = z_2^2) f^b(z_2^1) \varphi(j-1) \\ &\quad + \sum_{z_2^1=1}^{N^*} \sum_{z_2^2=1}^{N^*} \mathbb{P}(\varsigma(y_2) < L, Y_{\varsigma(y_2)}(y_2) = (z_2^1, 0)) f^a(z_2^2) \varphi(j+1) \end{aligned}$$

$$\begin{aligned}
& + \sum_{z_2^1=1}^{N^*} \sum_{z_2^2=1}^{N^*} \mathbb{P}(\varsigma(y_2) < L, Y_{\varsigma(y_2)}(y_2) = (0, z_2^2)) f^b(z_2^1) \varphi(j+1) \\
& = \varphi(j-1) \mathbb{P}(L < \varsigma(y_2)) + \varphi(j+1) \mathbb{P}(\varsigma(y_2) < L).
\end{aligned}$$

Since $\mathbb{P}(\varsigma(y) > t) \leq \mathbb{P}(\varsigma(z) > t)$ for $y = (y^1, y^2)$, $z = (z^1, z^2) \in \Omega_{N^*}^2$ with $z^1 \geq y^1$ and $z^2 \geq y^2$, for any $y_2 \in \Omega_{N^*}^2$,

$$\mathbb{P}(\varsigma((N^*, N^*)) > t) \geq \mathbb{P}(\varsigma(y_2) > t) \geq \mathbb{P}(\varsigma((1, 1)) > t).$$

Thus,

$$\begin{aligned}
\sum_{\hat{z} \in \Xi} P(\hat{y}, \hat{z}) \varphi(\hat{z}) & = \varphi(j-1) \mathbb{P}(N < \varsigma(y_2)) + \varphi(j+1) \mathbb{P}(\varsigma(y_2) < N) \\
& \leq \varphi(j-1)(1 - p_{N^*}) + \varphi(j+1)p_1.
\end{aligned}$$

From the previous expression, a sufficient condition for φ to be super-harmonic, is to satisfy the linear difference equation $p_1 \varphi(j+1) + (1 - p_{N^*}) \varphi(j-1) = \varphi(j)$, whose particular solution, satisfying the desired boundary conditions, is given by (12).

Proof of Lemma 2.5. Throughout, we set $\mathbf{1} = (1, 1)$ and $\mathbf{N}^* = (N^*, N^*)$. First, we will prove that the condition $\alpha > \mu + \theta$ implies an upper bound for p_1 . The independence of $\varsigma(\mathbf{1})$ and $L \sim \exp(2\alpha)$ implies that $\mathbb{P}(L > \varsigma(\mathbf{1})) = \int_0^\infty f_{\varsigma(\mathbf{1})}(t) e^{-2\alpha t} dt$, where $f_{\varsigma(\mathbf{1})}(t)$ is the probability density functions of $\varsigma(\mathbf{1})$. Using integration by parts,

$$\begin{aligned}
& \mathbb{P}(L > \varsigma(\mathbf{1})) \\
& = \int_0^\infty \frac{d}{dt} (-\mathbb{P}(\varsigma(\mathbf{1}) \geq t)) e^{-2\alpha t} dt = 1 - \int_0^\infty 2\alpha e^{-2\alpha t} \mathbb{P}(\varsigma(\mathbf{1}) \geq t) dt.
\end{aligned} \tag{46}$$

Let E_t be the event that there is neither a cancellation nor an arrival of market orders before time t at either side of the book. Since $\mathbb{P}(\varsigma(\mathbf{1}) \geq t) > \mathbb{P}(E_t) = e^{-2(\mu+\theta)t}$, by (46) and the assumption that $\alpha > \mu + \theta$,

$$\mathbb{P}(L > \varsigma(\mathbf{1})) = 1 - \int_0^\infty 2\alpha e^{-2\alpha t} \mathbb{P}(\varsigma(\mathbf{1}) \geq t) dt < 1 - \frac{\alpha}{\alpha + \mu + \theta} \leq \frac{1}{2}.$$

Thus, regardless of the sign of $p_1(1 - p_{N^*}^*)$, since $p_{N^*} < p_1 < \frac{1}{2}$, we have that $\lim_{j \rightarrow \infty} \varphi(j) = \infty$. $\square$

Proof of Lemma 2.6. We apply Theorem 2.2, for which we need to prove that $\pi(|f|) < \infty$ and $\pi(|g_t|) < \infty$. The latter assertions hold true if we can show that, for all $\hat{x} = (x_0, c_0, x_1, c_1) \in \Xi$,

$$\mathbb{E}(\tau_1 | \hat{x}) \leq C < \infty, \tag{47}$$

for a constant C , since

$$\begin{aligned}
\pi(|f|) & := \sum_{\hat{x} \in \Xi} \pi(\hat{x}) \mathbb{E}(\tau_1 | \hat{x}) \leq C \sum_{\hat{x} \in \Xi} \pi(\hat{x}) < \infty, \\
\pi(|g_t|) & := \sum_{\hat{x} \in \Xi} \pi(\hat{x}) \mathbb{P}(\tau_1 > t | \hat{x}) \leq \sum_{\hat{x} \in \Xi} \pi(\hat{x}) < \infty.
\end{aligned}$$

To show (47), we first need some notation. Let $\varsigma(x)$ be defined as in Lemma 2.4. Note that

$$\mathbb{E}(\tau_1 | (\tilde{x}_0, \zeta_0) = (x_0, c_0))$$

$$\leq \mathbb{E}(\tau_1 | (\tilde{x}_0, \zeta_0) = ((N^*, N^*), 1)) = \mathbb{E}(\zeta((N^*, N^*))) < \infty, \quad (48)$$

where the last inequality holds, since $\zeta((N^*, N^*)) \leq \min(\varpi_1, \varpi_2)$, where ϖ_i , $i = 1, 2$, is the hitting time at 0 of a 1-dimensional birth and death process with birth rate λ and death rate $\mu + \theta$ starting at N^* (for which is known the expectation is finite) and ϖ_1 is independent of ϖ_2 . Next, let $\mathcal{R}(x_0, c_0) = \{x_1 \in \Omega_{N^*}^2 : \mathbb{P}((\tilde{x}_1, \zeta_1) = (x_1, c_0 \pm 1) | (\tilde{x}_0, \zeta_0) = (x_0, c_0)) > 0\}$ and let

$$\begin{aligned} r_{x_1}^\pm((x_0, c_0)) &:= \mathbb{P}((\tilde{x}_1, \zeta_1) = (x_1, c_0 \pm 1) | (\tilde{x}_0, \zeta_0) = (x_0, c_0)), \quad c_0 > 1, \\ r_{x_1}((x_0, 1)) &:= \mathbb{P}((\tilde{x}_1, \zeta_1) = (x_1, 2) | (\tilde{x}_0, \zeta_0) = (x_0, 1)), \\ r_{\min}(x_0) &:= \min\{r_{x_1}^\pm((x_0, 2)) : x_1 \in \mathcal{R}(x_0, 2)\} \wedge \min\{r_{x_1}((x_0, 1)) : x_1 \in \mathcal{R}(x_0, 1)\}. \end{aligned}$$

Since, for any $c_0, c_1 > 1$, $r_{x_1}^\pm((x_0, c_0)) = r_{x_1}^\pm((x_0, c_1))$, it follows that $0 < r_{\min}(x_0) \leq r_{x_1}^\pm((x_0, c_0))$ for all $c_0 \in \{2, 3, \dots\}$ and $x_1 \in \mathcal{R}(x_0, c_0)$. Therefore,

$$\begin{aligned} r_{\min}(x_0) &\sum_{x_1 \in \mathcal{R}(x_0, c_0)} \mathbb{E}(\tau_1 | x_0, c_0, x_1, c_0 \pm 1) \\ &\leq \mathbb{E}(\tau_1 | (\tilde{x}_0, \zeta_0) = (x_0, c_0)) < \mathbb{E}(\zeta((N^*, N^*))) < \infty. \end{aligned}$$

This implies (47), which in turn implies the result as explained above.

Proof of Theorem 2.11. Since the state space, Λ , is countable, every finite subset of the state space is an atom (e.g., see [14, Chapter 5, pg 105]) and, hence, we are able to construct explicitly the solution of the Poisson equation (22). Indeed, by Equation (17.38) in [14] and the discussion therein, for $C_1 := \{\bar{z} = (x, c, u) \in \Lambda : x \in \Omega_{N^*}^2, c = 1, u \in \{-1/2, 1/2\}\}$, we have that

$$\hat{h}(\bar{z}) = \mathbb{E}_{\bar{z}} \left[\sum_{k=1}^{\sigma_{C_1}} \bar{h}(V_k) \right], \quad (49)$$

where $\sigma_{C_1} = \min\{n \geq 0 \mid V_n \in C_1\}$. Since for any $\bar{z} \in \Lambda$, $|h(\bar{z})| \leq 1/2$, $|\bar{h}(\bar{z})| \leq 1$ and, thus, $|\hat{h}(\bar{z})| \leq \mathbb{E}_{\bar{z}}(\sigma_{C_1})$. Therefore, to conclude that the invariance principle (26) holds true, it suffices to show that

$$\nu \left(\mathbb{E}^2(\sigma_{C_1}) \right) := \sum_{\bar{z} \in \Lambda} \nu(\bar{z}) \mathbb{E}_{\bar{z}}^2(\sigma_{C_1}) < \infty. \quad (50)$$

Let $C_j = \{\bar{z} = (x, c, u) \in \Lambda \mid x \in \Omega_{N^*}^2, c = j, u \in \{-1/2, 1/2\}\}$. Each C_j is finite and $\{C_j\}_{j \geq 1}$ forms a partition of Λ . Clearly, for every n , if $V_n \in C_j$, with $j \geq 2$, then $V_{n+1} \in \{C_{j-1}, C_{j+1}\}$. Moreover, with the notation of Lemma 2.4 and, as proved in the proof of Lemma 2.5, for any $\bar{z} = (x, c, u) \in C_j$

$$\mathbb{P}(V_{n+1} \in C_{j+1} | V_n = \bar{z}) = \mathbb{P}(\zeta(x) < L) \leq \mathbb{P}(\zeta(\mathbf{1}) < L) = p_1.$$

Consider now a birth and death process $\tilde{V}_n \in \mathbb{N}$ with birth probability p_1 and death probability $1 - p_1$, and note that

$$\mathbb{P}[V_{n+1} \in C_{j-1} \mid V_n = \bar{z}] \geq 1 - p_1 = \mathbb{P}[\tilde{V}_{n+1} = j - 1 \mid \tilde{V}_n = j].$$

Denote by $\sigma_1^{\tilde{V}}$ the first hitting time of $\tilde{V}$ to the point 1. That is, $\sigma_1^{\tilde{V}} = \min\{n > 0 \mid \tilde{V}_n = 1\}$. Then, since V dies more frequently than $\tilde{V}$, for any $\bar{z} \in C_j$,

$$\mathbb{E}_{\bar{z}}[\sigma_{C_1}] \leq \mathbb{E}_j[\sigma_1^{\tilde{V}}] = \frac{j-1}{1-2p_{N^*}}, \quad (51)$$

where, in the last equality we use that $p_1 < 1/2$ (see [18, Section 3.1]).

The next step is to bound the terms $v(C_j)$ for $j \geq 2$. To shorten notation, define $\Theta := \Omega_{N^*} \times \{-1/2, 1/2\}$. Recall that if the spread is larger than one, the spread will widen or shrink right after every price change, whereas if the spread is 1, it will surely widen at the next step. Thus, for any $\bar{y} \in C_1$ and any $\bar{z} \in C_j$ with $j \geq 2$, $P^{ext}[\bar{y}, C_2] = 1$ and $P^{ext}[\bar{z}, C_{j+1}] + P^{ext}[\bar{z}, C_{j-1}] = 1$. By the definition of a stationary measure,

$$v(C_1) = \int_{\Lambda} P^{ext}[\bar{z}, C_1] v(d\bar{z}) = \sum_{\bar{z} \in C_2} P^{ext}[\bar{z}, C_1] v(\bar{z}) = \sum_{\bar{z} \in C_2} (1 - P^{ext}[\bar{z}, C_3]) v(\bar{z}),$$

which implies that

$$\sum_{\bar{z} \in C_2} P^{ext}[\bar{z}, C_3] v(\bar{z}) = v(C_2) - v(C_1). \quad (52)$$

Analogously,

$$\begin{aligned} v(C_2) &= \int_{\Lambda} P^{ext}[\bar{z}, C_2] v(d\bar{z}) = \sum_{\bar{z} \in C_1} P^{ext}[\bar{z}, C_2] v(\bar{z}) + \sum_{\bar{z} \in C_3} P^{ext}[\bar{z}, C_2] v(\bar{z}) \\ &= v(C_1) + \sum_{\bar{z} \in C_3} (1 - P^{ext}[\bar{z}, C_4]) v(\bar{z}), \end{aligned}$$

which implies that

$$\sum_{\bar{z} \in C_3} P^{ext}[\bar{z}, C_4] v(\bar{z}) = v(C_1) - v(C_2) + v(C_3). \quad (53)$$

Next, note that, for $j > 2$,

$$\begin{aligned} v(C_{j-1}) &= \int_{\Lambda} P^{ext}[\bar{z}, C_{j-1}] v(d\bar{z}) \\ &= \sum_{\bar{z} \in C_{j-2}} P^{ext}[\bar{z}, C_{j-1}] v(\bar{z}) + \sum_{\bar{z} \in C_j} (1 - P^{ext}[\bar{z}, C_{j+1}]) v(\bar{z}) \\ &= v(C_j) + \sum_{\bar{z} \in C_{j-2}} P^{ext}[\bar{z}, C_{j-1}] v(\bar{z}) - \sum_{\bar{z} \in C_j} P^{ext}[\bar{z}, C_{j+1}] v(\bar{z}). \end{aligned} \quad (54)$$

Therefore,

$$\sum_{\bar{z} \in C_j} P^{ext}[\bar{z}, C_{j+1}] v(\bar{z}) = v(C_j) - v(C_{j-1}) + \sum_{\bar{z} \in C_{j-2}} P^{ext}[\bar{z}, C_{j-1}] v(\bar{z}). \quad (55)$$

Applying the previous equation recursively, for all even $j \geq 2$,

$$\begin{aligned} \sum_{\bar{z} \in C_j} P^{ext}[\bar{z}, C_{j+1}]v(\bar{z}) \\ = v(C_j) - v(C_{j-1}) + v(C_{j-2}) - v(C_{j-3}) + \cdots + v(C_4) - v(C_3) \\ + \sum_{\bar{z} \in C_2} P^{ext}[\bar{z}, C_3]v(\bar{z}), \end{aligned}$$

and, thus, by (52),

$$\begin{aligned} \sum_{\bar{z} \in C_j} P^{ext}[\bar{z}, C_{j+1}]v(\bar{z}) = v(C_j) - v(C_{j-1}) + v(C_{j-2}) - v(C_{j-3}) + \cdots \\ + v(C_4) - v(C_3) + v(C_2) - v(C_1). \end{aligned} \quad (56)$$

However, if $j \geq$ is odd, by (53),

$$\begin{aligned} \sum_{\bar{z} \in C_j} P^{ext}[\bar{z}, C_{j+1}]v(\bar{z}) = v(C_j) - v(C_{j-1}) + v(C_{j-2}) - v(C_{j-3}) + \cdots \\ + v(C_5) - v(C_4) + \sum_{\bar{z} \in C_3} P^{ext}[\bar{z}, C_4]v(\bar{z}) \\ = v(C_j) - v(C_{j-1}) + v(C_{j-2}) - v(C_{j-3}) + \cdots \\ + v(C_5) - v(C_4) + v(C_1) - v(C_2) + v(C_3). \end{aligned} \quad (57)$$

Eqs. (56)–(57) imply that

$$v(C_{j+1}) = \sum_{\bar{z} \in C_j} P^{ext}[\bar{z}, C_{j+1}]v(\bar{z}) + \sum_{\bar{z} \in C_{j+1}} P^{ext}[\bar{z}, C_{j+2}]v(\bar{z}).$$

However, by the definition of a stationary measure,

$$v(C_{j+1}) = \sum_{\bar{z} \in C_j} P^{ext}[\bar{z}, C_{j+1}]v(\bar{z}) + \sum_{\bar{z} \in C_{j+2}} P^{ext}[\bar{z}, C_{j+1}]v(\bar{z}).$$

The previous two equations yield the following relation,³

$$\sum_{\bar{z} \in C_{j+1}} P^{ext}[\bar{z}, C_{j+2}]v(\bar{z}) = \sum_{\bar{z} \in C_{j+2}} P^{ext}[\bar{z}, C_{j+1}]v(\bar{z}). \quad (58)$$

We are now ready to bound the term $v(C_j)$. To that end, notice that for any $\bar{z} \in C_j$, $P^{ext}[\bar{z}, C_{j-1}] \geq (1 - p_1)$ and, thus,

$$\begin{aligned} v(C_j) &= \sum_{\bar{z} \in C_{j-1}} P^{ext}[\bar{z}, C_j]v(\bar{z}) + \sum_{\bar{z} \in C_{j+1}} P^{ext}[\bar{z}, C_j]v(\bar{z}) \\ &= \sum_{\bar{z} \in C_j} P^{ext}[\bar{z}, C_{j-1}]v(\bar{z}) + \sum_{\bar{z} \in C_{j+1}} P^{ext}[\bar{z}, C_j]v(\bar{z}) \\ &\geq (1 - p_1)v(C_j) + (1 - p_1)v(C_{j+1}). \end{aligned}$$

³ Eq. (58) gives some insight into the structure of the stationary measure v and can be regarded as a “batch” version of the so-called Detailed Balance Conditions for Markov Chains, which are important for analyzing reversible processes (cf. [11, Chapter 1.2]).

Therefore, $v(C_{j+1}) \leq v(C_j)p_1/(1 - p_1)$, which, by induction, implies that

$$v(C_{j+1}) \leq \left(\frac{p_1}{1 - p_1}\right)^{j-1} v(C_2). \quad (59)$$

Finally, by (51) and (59) and the fact that $p^* = p_1/(1 - p_1) < 1$, for some bounded constant C^* ,

$$\begin{aligned} v(\mathbb{E}^2(\sigma_{C_1})) &= \sum_{\bar{z} \in C_1} v(\bar{z}) \mathbb{E}_{\bar{z}}^2(\sigma_{C_1}) + \sum_{j=2}^{\infty} \sum_{\bar{z} \in C_j} v(\bar{z}) \mathbb{E}_{\bar{z}}^2(\sigma_{C_1}) \\ &= C^* + \sum_{j=2}^{\infty} v(C_j) \left(\frac{j-1}{1 - 2p_{N^*}}\right)^2 \\ &\leq C^* + v(C_1) \sum_{j=2}^{\infty} (p^*)^{j-1} \left(\frac{j-1}{1 - 2p_{N^*}}\right)^2 < \infty. \end{aligned} \quad (60)$$

It only remains to show that the variance in the FCLT can be written as (27). But, by Theorem 17.5.3 in [14], it is enough to show that the Markov chain $\{V_n\}_{n \geq 1}$ is ergodic and there exists a function $F : \Lambda \rightarrow [0, \infty]$ such that $v(F^2) < \infty$ and

$$\Delta F(\bar{z}) \leq -1 + b \mathbb{1}_{\{\bar{z} \in B\}}, \quad (61)$$

for a constant $b < \infty$ and a finite set B , where Δ is the operator $\Delta F(\bar{z}) := \mathbb{E}_{\bar{z}}[F(V_1) - F(V_0)]$ (cf. Section 14.2.1 in [14]). However, it is known (cf. Section 13.1.2 [14]) that aperiodic positive Harris chains over a countable state space are ergodic and by Proposition 14.1.2 and Theorem 14.2.3(ii) therein, the function $F(\bar{z}) := \mathbb{E}_{\bar{z}}[\sigma_B]$ satisfies (61), where σ_B is the first hitting time of the set B in (61). Since it was proven above that $v(\mathbb{E}^2[\sigma_{C_1}]) < \infty$, we take $B = C_1$ to conclude the proof. $\square$

Proof of Lemma 3.1. Let us start by noting that the generator of the two-dimensional random walk Y defined in (30) is given by the finite difference operator $\mathcal{L}$ defined in (32). More concretely, for a function $\phi : \bar{\Omega}_{N^*}^2 \rightarrow \mathbb{R}$, $\mathcal{L}\phi(x, y)$ is defined analogously to (32) but replacing $u(t, x, y)$ with $\phi(x, y)$. For simplicity, we denote $\varsigma := \varsigma(x, y)$ and remark that ς is an absolutely continuous random variable. Let $\bar{u}(t, x, y)$ be an arbitrary bounded function such that $t \mapsto \bar{u}(t, x, y)$ is C_1 for all (x, y) and $(t, x, y) \mapsto \partial_t \bar{u}(t, x, y)$ is bounded. Fix $T > 0$ and let $f(t, x, y) = \bar{u}(T - t, x, y)$. Under the stated conditions, $\bar{u}$ belongs to the domain of the generator $\mathcal{L}$ and, thus, the process

$$f(t, X_t) - \int_0^t \left(\frac{\partial}{\partial r} + \mathcal{L} \right) f(r, Y_r) dr, \quad t \in [0, T]$$

is a local martingale. Therefore,

$$M_t := \bar{u}(T - t, Y_t) - \int_0^t \left(-\frac{\partial}{\partial t} + \mathcal{L} \right) \bar{u}(T - r, Y_r) dr, \quad t \in [0, T],$$

is a martingale. Let $\sigma := T \wedge \varsigma$. By the Optional Sampling Theorem,

$$\bar{u}(T, x, y) = \mathbb{E}[\bar{u}(T - \sigma, Y_\sigma)] - \mathbb{E} \left[\int_0^\sigma \left(-\frac{\partial}{\partial t} + \mathcal{L} \right) \bar{u}(T - r, Y_r) dr \right]. \quad (62)$$

Now, suppose that $\bar{u}(t, x, y)$ solves the following initial value problem,

$$\begin{cases} \left(-\frac{\partial}{\partial t} + \mathcal{L}\right) \bar{u}(T-r, x, y) = 0 & \text{for } 0 \leq r \leq T, (x, y) \in \{1, 2, \dots, N^*\}^2. \\ \bar{u}(T-r, x, y) = \mathbb{1}_{\{(x,y)=\bar{a}\}} & \text{for } 0 \leq r \leq T, (x, y) \in \mathcal{A}. \\ \bar{u}(0, x, y) = \mathbb{1}_{\{(x,y)=\bar{a}\}} & \text{for } (x, y) \in \{0, 1, 2, \dots, N^*\}^2. \end{cases} \quad (63)$$

In that case, by Eq. (62),

$$\begin{aligned} \bar{u}(T, x, y) &= \mathbb{E}[\bar{u}(T-\sigma, Y_\sigma)] \\ &= \mathbb{E}[\bar{u}(T-\sigma, Y_\sigma) \mathbb{1}_{\{\sigma < T\}} + \bar{u}(T-\sigma, Y_\sigma) \mathbb{1}_{\{\sigma = T\}}] \\ &= \mathbb{E}[\mathbb{1}_{\{Y_\sigma = \bar{a}\}} \mathbb{1}_{\{\sigma < T\}} + \bar{u}(0, Y_\sigma) \mathbb{1}_{\{\sigma = T\}}] \\ &= \mathbb{P}[Y_\zeta = \bar{a}, \zeta \leq T], \end{aligned}$$

which implies that $\bar{u}(t, x, y) = u_{\bar{a}}(t, x, y)$. $\square$

Proof of Proposition 3.2. Let

$$w(t, x, y) = \left(\frac{\lambda}{v}\right)^{\frac{\bar{a}_1 + \bar{a}_2}{2}} e^{t(2(\lambda+v) - 4\sqrt{\lambda v})} \mathbb{1}_{\{(x,y)=\bar{a}\}}.$$

Fix $\tilde{v}(t, x, y) = v(t, x, y) - w(t, x, y)$ and note that $v(t, x, y)$ satisfies the system (66) if and only if $\tilde{v}$ is a solution to the initial value problem:

$$\begin{cases} -\left(\frac{\partial}{\partial t} - \sqrt{\lambda v} \Delta\right) \tilde{v}(t, x, y) = \left(\frac{\partial}{\partial t} - \sqrt{\lambda v} \Delta\right) w(t, x, y) & \text{for } t \geq 0, (x, y) \in \Omega_{N^*} \\ \tilde{v}(t, x, y) = 0 & \text{for } t \geq 0, (x, y) \in \mathcal{A} \\ \tilde{v}(0, x, y) = 0 & \text{for } (x, y) \in \bar{\Omega}_{N^*}. \end{cases} \quad (64)$$

Let $\{\psi_k(t)\}_{k=1}^{N^{*2}}$ and $\{\varsigma_k^{\bar{a}}\}_{k=1}^{N^{*2}}$ be such that $\tilde{v}(t, x, y) = \sum_k^{N^{*2}} \psi_k(t) f_k(x, y)$ and $\mathbb{1}_{\{(x,y)=\bar{a}+1\}} = \sum_{k=1}^{N^{*2}} \varsigma_k^{\bar{a}} f_k(x, y)$. Using the first representation, the left-hand side of the first equation in (64) becomes

$$\begin{aligned} -\left(\frac{\partial}{\partial t} - \sqrt{\lambda v} \Delta\right) \tilde{v}(t, x, y) &= -\sum_{k=1}^{N^{*2}} \psi'_k(t) f_k(x, y) - \psi_k(t) \sqrt{\lambda v} \Delta f_k(x, y) \\ &= \sum_{k=1}^{N^{*2}} \left[-\psi'_k(t) + \psi_k(t) \sqrt{\lambda v} \xi_k\right] f_k(x, y). \end{aligned}$$

Similarly, the right-hand side of the first equation in (64) is given by

$$\begin{aligned} \left(\frac{\partial w}{\partial t} - \sqrt{\lambda v} \Delta\right) w &= -\left(\frac{\lambda}{v}\right)^{\frac{\bar{a}_1 + \bar{a}_2}{2}} \sqrt{\lambda v} e^{t(2(\lambda+v) - 4\sqrt{\lambda v})} \mathbb{1}_{\{(x,y)=\bar{a}+1\}} \\ &= -e^{t(2(\lambda+v) - 4\sqrt{\lambda v})} \left(\frac{\lambda}{v}\right)^{\frac{\bar{a}_1 + \bar{a}_2}{2}} \sqrt{\lambda v} \sum_{k=1}^{N^{*2}} \varsigma_k^{\bar{a}} f_k(x, y). \end{aligned}$$

Combining the previous two expressions and recalling that $\{f_k(x, y)\}_k$ is an orthonormal basis, it follows that the function $\tilde{v}(t, x, y) = \sum_k \psi_k(t) f_k(x, y)$ will solve the system (64) if and only

if, for every k , the function $\psi_k(t)$ satisfies the following equation:

$$-\psi'_k(t) + \psi_k(t)\sqrt{\lambda v}\xi_k = -e^{t(2(\lambda+v)-4\sqrt{\lambda v})}\left(\frac{\lambda}{v}\right)^{\frac{\bar{a}_1+\bar{a}_2}{2}}\sqrt{\lambda v}\zeta_k^{\bar{a}}, \quad (65)$$

with the initial condition $\psi_k(0) = 0$. It is easy to see that the previous differential equation is well posed and has solution

$$\psi_k(t) = \left(\frac{\lambda}{v}\right)^{\frac{\bar{a}_1+\bar{a}_2}{2}} \frac{\sqrt{\lambda v}\zeta_k^{\bar{a}}}{2(\lambda+v) - \sqrt{\lambda v}(4+\xi_k)} \left[e^{t\sqrt{\lambda v}\xi_k} - e^{t(2(\lambda+v)-4\sqrt{\lambda v})} \right].$$

Therefore,

$$\begin{aligned} \tilde{v}(t, x, y) &= \left(\frac{\lambda}{v}\right)^{\frac{\bar{a}_1+\bar{a}_2}{2}} \sum_k \frac{\sqrt{\lambda v}\zeta_k^{\bar{a}}}{2(\lambda+v) - \sqrt{\lambda v}(4+\xi_k)} \left[e^{t\sqrt{\lambda v}\xi_k} - e^{t(2(\lambda+v)-4\sqrt{\lambda v})} \right] f_k(x, y), \end{aligned}$$

satisfies the initial value problem (64), which in turn, implies that

$$\begin{aligned} u(t, x, y) &= \left(\frac{\lambda}{v}\right)^{\frac{\bar{a}_1+\bar{a}_2-x-y}{2}} \left[\sum_{k=1}^{N^*2} \frac{\sqrt{\lambda v}\zeta_k^{\bar{a}}}{2(\lambda+v) - \sqrt{\lambda v}(4+\xi_k)} \left(1 - e^{-t(2(\lambda+v)-(4+\xi_k)\sqrt{\lambda v})} \right) \right. \\ &\quad \left. \times f_k(x, y) + \mathbb{1}_{\{(x,y)=\bar{a}\}} \right], \end{aligned}$$

is a solution of (31). Then, the representation (35) immediately follows by noting that $\zeta_k^{\bar{a}} = f_k(\bar{a}+1)$ and rewriting the previous expression in terms of $\chi = \lambda/v$.

Lemma A.1. A function $u : [0, T] \times \bar{\Omega}_{N^*}^2 \rightarrow \mathbb{R}$ is a solution of the system of differential equations (31) if and only if the function $v(t, x, y)$ defined by

$$v(t, x, y) = \left(\frac{\lambda}{v}\right)^{\frac{x+y}{2}} e^{t(2(\lambda+v)-4\sqrt{\lambda v})} u(t, x, y),$$

solves the system of difference equations

$$\begin{cases} \left(-\frac{\partial}{\partial t} + \sqrt{\lambda v}\Delta\right)v(t, x, y) = 0, & \text{for } t \geq 0, (x, y) \in \Omega_{N^*}^2, \\ v(t, x, y) = \left(\frac{\lambda}{v}\right)^{\frac{\bar{a}_1+\bar{a}_2}{2}} e^{t(2(\lambda+v)-4\sqrt{\lambda v})} \mathbb{1}_{\{(x,y)=\bar{a}\}}, & \text{for } t \geq 0, (x, y) \in \mathcal{A}, \\ v(0, x, y) = \left(\frac{\lambda}{v}\right)^{\frac{\bar{a}_1+\bar{a}_2}{2}} \mathbb{1}_{\{(x,y)=\bar{a}\}}, & \text{for } (x, y) \in \bar{\Omega}_{N^*}^2, \end{cases} \quad (66)$$

where $\bar{a} = (\bar{a}_1, \bar{a}_2) \in \mathcal{A}$ and, for each fixed t , $\Delta v(t, x, y)$ is defined as in (34) with respect to x and y .

Proof. The proof is standard and is omitted. $\square$

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